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Fast Estimation of True Bounds on Bermudan Option Prices Under Jump-diffusion Processes

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Fast pricing of American-style options has been a difficult problem since it was first introduced to the financial markets in 1970s, especially when the underlying stocks' prices follow some jump-diffusion processes. In this paper, we extend the "true martingale algorithm" proposed by Belomestny *et al.* (2009) for the pure-diffusion models to the jump-diffusion models, to fast compute true tight upper bounds on the Bermudan option price in a non-nested simulation manner. By exploiting the martingale representation theorem on the optimal dual martingale driven by jump-diffusion processes, we are able to explore the unique structure of the optimal dual martingale and construct an approximation that preserves the martingale property. The resulting upper bound estimator avoids the nested Monte Carlo simulation suffered by the original primal-dual algorithm, therefore significantly improving the computational efficiency. Theoretical analysis is provided to guarantee the quality of the martingale approximation. Numerical experiments are conducted to verify the efficiency of our algorithm.

Keywords: Bermudan option; Jump-diffusion processes; Optimal dual martingale; Martingale property; Monte Carlo simulation.

JEL Classification: C,G

1. Introduction

Pricing American-style derivatives (which is essentially an optimal stopping problem) has been an active and challenging problem in the last thirty years, especially when the underlying stocks' prices follow some jump-diffusion processes, as they become more and more critical to investors. To present time, various jump-diffusion models for financial modelling have been proposed to fit the real data in financial markets, including: (i) the normal jump-diffusion model, see Merton (1976); (ii) the affine jump-diffusion models, see Duffie *et al.* (2000); (iii) the jump models based on Levy processes, see Cont and Tankov (2003); (iv) the double exponential, mixed-exponential and hyper-exponential jump-diffusion models, see Kou (2002), Cai and Kou (2011), and Cai and Kou (2012). All these models are trying to capture some interesting features of the market behaviour that cannot be well explained by pure-diffusion models, such as the heavy-tail risk suffered by the market. In general, closed-form expressions for the American-style derivatives can hardly be derived under these jump-diffusion models due to the multiple exercise opportunities and the randomness in the underlying asset price caused by both jumps and diffusions. Hence, various numerical methods have been proposed to tackle the American-style option pricing problems under the jump-diffusion models, including: (i) solving the free boundary problems via lattice or differential equation methods, see Amin (1993), Këllezi and Webber (2004), Feng and Linetsky (2008), Fang and Oosterlee (2009), Feng and Lin (2013); (ii) quadratic approximation and piece-wise exponential approximation methods, see Pham (1997), Gukhal (2001), Kou and Wang (2004). A thorough study on jump-diffusion models for asset pricing has been done by Kou (2008). More broadly, an elegant overview of financial models under jump processes is provided in Cont and Tankov (2003).

Another class of widely-used methods are based on Monte Carlo simulation, and they have been successfully implemented on Bermudan option pricing problems under the pure-diffusion models, see Bossaerts (1989), Tilley (1993), Longstaff and Schwartz (2001), Tsitsiklis and van Roy (2001). In particular, Longstaff and Schwartz (2001), Tsitsiklis and van Roy (2001) propose to approximate the continuation values by regression on certain sets of basis functions (called "function bases"), which leads to good suboptimal exercise strategies and lower bounds on the exact option price. Moreover, their methods bypass "the curse of dimensionality" and scale well with the number of underlying variables, working efficiently for high-dimensional problems under the pure-diffusion models. Though these methods can be naturally adapted to option pricing problems under the jump-diffusion setting, two key questions regarding the effectiveness of these methods remain to be addressed: (i) how to choose the function bases for regression; (ii) how to measure the quality of the lower bounds.

The second question is partially addressed by the dual approach proposed by Rogers (2002), Haugh and Kogan (2004), and Anderson and Broadie (2004). They are able to generate the upper bounds on the option price by solving the associated dual problem, which is obtained by subtracting the payoff function by a dual martingale adapted to a proper filtration. In theory, if the dual martingale is the Doob-Meyer martingale part of the option price process, namely the "optimal dual martingale", then the resulting upper bound equals the exact option price. In practice, the optimal dual martingale is not available, but good approximations of it can generate tight upper bounds. With the access to the upper bounds, the quality of suboptimal exercise strategies or lower bounds could be measured empirically by looking at the duality gaps, which are the differences between the lower bounds and the upper bounds. A multiplicative version of dual approach based on multiplicative Doob-Meyer decomposition is proposed by Jamshidian (2007). A thorough comparison between the additive dual approach and the multiplicative dual approach can be found in Chen and Glasserman (2007). Glasserman (2004) provides an elegant and thorough overview on the duality theory for option pricing problems.

A lot work has been done following the duality theory. To name a few, Bender (2011), Chandramouli and Haugh (2012), Balder *et al.* (2013) and Bender *et al.* (2013) develop the multilevel primal-dual approach for optimal stopping problems with multiple stopping times. Belomestny *et al.* (2013) optimize the cost of simulation by considering a multilevel Monte Carlo technique for the primal-dual approach. Desai *et al.* (2012) consider an additional path-wise optimization procedure in constructing the dual martingales for optimal stopping problems. Schoenmakers *et al.* (2013) provide a beautiful characterization of the almost surely optimal dual martingales in the context of dual valuation of Bermudan options and outline the development of new algorithms. In the context of solving optimal stopping problem in both discrete and continuous time setting, Belomestny (2013) considers the optimization of a penalized dual objective functional over a class of dual martingales without involving nested simulation. Ye and Zhou (2013b) apply the primal-dual approach with particle filtering techniques to optimal stopping problems of partially observable Markov processes. Rogers (2007), Brown *et al.* (2010) generalize the duality theory to general discrete-time dynamic programming problems and provide a broader interpretation of the dual martingale. From Brown *et al.* (2010)'s perspective, the dual martingale can be regarded as the penalty for the access to the future information (information relaxation) and different degrees of relaxation result in different levels of upper bounds. In particular, the dual martingales constructed by Haugh and Kogan (2004), Anderson and Broadie (2004) can be interpreted as perfect information relaxation, which means the option holder has access to all the future prices of the underlying assets. Ye and Zhou (2012) consider a parameterized path-wise optimization technique in constructing the penalties for general dynamic programming problems. Ye and Zhou (2013a) also develop the duality theory for general dynamic programming problems under a continuous-time setting.

The numerical effectiveness of the primal-dual approach has been demonstrated in pricing multi-dimensional American-style options. The algorithm generates good suboptimal exercise strategies and good lower-upper bound pairs with sufficiently small duality gaps. A possible deficiency of the algorithm is the heavy computation (quadratic in the number of exercisable periods), caused by the nested simulation in constructing the dual martingale. To address the computational issue, Belomestny *et al.* (2009) propose an alternative algorithm to generate approximations of the optimal dual martingale via non-nested simulation under the Wiener process setting. By exploiting the martingale representation theorem on the optimal dual martingale driven by Wiener processes, they are able to approximate the optimal dual martingale through regressing the integrand on some function bases at a finite number of time points. The resulting approximation preserves the martingale property and generates a true upper bound on the option price. More importantly, their algorithm avoids nested simulation and is linear in the number of exercisable periods.

In this paper, we will generalize Belomestny *et al.* (2009)'s idea of "true martingale" to Bermudan option pricing problems under jump-diffusion processes and provide a new perspective in understanding the structure of the optimal dual martingale, which facilitates us to construct good approximations of it. To the best of our knowledge, we are among the first to ever consider estimating the upper bounds on American-style option price under the jump-diffusion models. In a greater detail, we have made the following contributions.

- Under the jump-diffusion setting, we explore the structure of the optimal dual martingales in the dual formulations of both the Bermudan and American-style option prices (Theorem 3.1 and Theorem 3.2), which is the underpinning of our proposed approach to generating tight upper bounds.
- We extend the "true martingale algorithm" proposed by Belomestny *et al.* (2009) under pure-diffusion setting to jump-diffusion setting (referred to as T-M algorithm) to fast compute tight upper bounds on the Bermudan option price under jump-diffusion models. The resulting approximation (called "true martingale approximation") preserves the martingale property, and therefore generates true upper bounds on the Bermudan option price. Moreover, compared with the primal-dual algorithm proposed by Anderson and Broadie (2004) (A-B algorithm), our T-M algorithm avoids the nested Monte Carlo simulation and scales linearly with the exercisable periods, and hence achieves a higher computational efficiency.
- We prove that the mean square error between the true martingale approximation and the objective martingale converges to zero as the time and jump space partitions both go to zero (Theorem 3.4).
- We investigate the numerical effectiveness of Longstaff and Schwartz (2001)'s least-squares regression approach (L-S algorithm) for Bermudan option price under the jump-diffusion models. In particular, we find that by incorporating the European option price under the corresponding pure-diffusion model (referred to as the "non-jump European option") into the function basis of the L-S algorithm, the quality of the induced suboptimal exercise strategies and the lower bounds can be further improved.
- Motivated by the explicit structure of the optimal dual martingale (Theorem 3.2), we propose simple yet powerful function bases that can be employed in the T-M algorithm to obtain tight upper bounds on the option price. By implementing our algorithm and the A-B algorithm on several sets of numerical experiments, the numerical results demonstrate that both methods can generate tight and stable upper bounds on option price. However, we observe that our algorithm is much more efficient than the A-B algorithm in practice due to the relief from nested simulation.

To summarize, the rest of this paper will be organized as follows. In Section 2, we describe the Bermudan option pricing problems under general jump-diffusion models and review the dual approach. We present the extension of true martingale approach to jump-diffusion setting and provide its error analysis in Section 3. Section 4 focuses on the detailed T-M algorithm and its numerical features. Numerical experiments are conducted in Section 5 to verify the computational efficiency of the T-M algorithm. Conclusion and future directions are given in Section 6.

2. Model formulation

2.1 *Preliminaries*

In this subsection, we will introduce some standard definitions in jump processes that will appear throughout the paper. They can be found in Cont and Tankov (2003).

Definition 2.1: [**Poisson random measure**] (Definition 2.18 in Cont and Tankov (2003)) Let $(\Omega, \mathcal{F}, \mathrm{P})$ be a probability space, $G \subset \mathrm{R}^{d+1}$ and μ be a given (positive) Radon measure on $(G, \mathcal{G})$. A Poisson random measure on G with intensity μ is an integer-valued random measure:

$$\begin{aligned} \mathcal{P} :& \Omega \times \mathcal{G} \to \mathrm{N} \\ & (\omega, A) \mapsto \mathcal{P}\,(\omega, A) \end{aligned}$$

such that: (i) for (almost all) $\omega \in \Omega, \mathcal{P}(\omega, \cdot)$ is an integer-valued Radon measure on G; (ii) for each measurable set $A \in \mathcal{G}$, $\mathcal{P}(\cdot, A)$ is a Poisson random variable with parameter $\mu(A)$:

$$Pr\Big(\mathcal{P}\,(\cdot, A) = k\Big) = e^{-\mu(A)} \frac{(\mu(A))^k}{k!}, \quad \forall\, k \in \mathrm{N};$$

(iii) for disjoint measurable sets $A_1, ..., A_n \in \mathcal{G}$, the variables $\mathcal{P}(\cdot, A_1), ..., \mathcal{P}(\cdot, A_n)$ are independent.

To parallel with the Wiener measure, we further define the associated compensated Poisson random measure as follows.

Definition 2.2: [**compensated Poisson random measure**] Assuming $\mathcal{P}(\cdot, \cdot)$ is a Poisson random measure with the intensity Radon measure $\mu(\cdot)$, then the corresponding compensated Poisson random measure can be constructed by subtracting $\mathcal{P}(\cdot, \cdot)$ by its intensity measure:

$$\begin{aligned} \tilde{\mathcal{P}} :& \Omega \times \mathcal{G} \to \mathrm{R} \\ & (\omega, A) \mapsto \tilde{\mathcal{P}}\,(\omega, A) = \mathcal{P}(\omega, A) - \mu(A). \end{aligned}$$

From Definition 2.2, one can easily obtain that, for $A \in \mathcal{G}$, $E[\tilde{\mathcal{P}}(\cdot, A)] = 0$ and $Var[\tilde{\mathcal{P}}(\cdot, A)] = Var[\mathcal{P}(\cdot, A)] = \mu(A)$. Therefore, we call $\tilde{\mathcal{P}}(\cdot, A)$ a compensated Poisson random variable. Clearly, compensated Poisson random variables parallel to normal random variables with mean zero. Furthermore, to connect Poisson random measure with jump processes, we summarize some results obtained by Cont and Tankov (2003) in the following Theorem 2.3.

Theorem 2.3: *Suppose a Poisson random measure $\mathcal{P}(ds, dy)$ on $G = [0, T] \times \mathrm{R}^d$ with the intensity measure $\mu(ds \times dy)$ is described as the counting measure associated with a random configuration of points $(T_n, Y_n) \in G$:*

$$\mathcal{P} = \sum_{n \geq 1} \delta_{(T_n, Y_n)},$$

where $(T_n(\omega), Y_n(\omega)) \in [0, T] \times \mathrm{R}^d$ corresponds to an observation made at time $T_n(\omega)$ and de-

scribed by a random variable $Y_n(\omega)$. $f(s,y)$ *is a* μ*-measurable function. Then*

$$X(t) \quad = \int_0^t \int_{\mathrm{R}^d} f(s,y)\mathcal{P}(ds,dy), \quad 0 \le t \le T,$$

is a jump process whose jumps happen at the random times T_n *and have amplitudes given by* $f(T_n, Y_n)$. *Furthermore, the corresponding compensated jump process*

$$\tilde{X}(t) \quad = \int_0^t \int_{\mathrm{R}^d} f(s,y)\tilde{\mathcal{P}}(ds,dy), \quad 0 \le t \le T,$$

is a martingale adapted to the filtration generated by $\mathcal{P}$.

Basically, Definitions 2.1 and 2.2, Theorem 2.3 summarize the basic properties possessed by Poisson random measure, and characterize the close connection between Poisson random measures and jump processes. They provide an intuitive understanding towards the construction of the Poisson random measures induced by jump processes. Overall, the compensated Poisson random measure possesses zero-mean and independent increment properties, which are the two fundamental properties in deriving the martingale representation theorem driven by Poisson random measures. With these useful tools, now we can formally describe the Bermudan option pricing problem under a general jump-diffusion model.

2.2 *Primal problem*

In this paper, we consider a special case of asset price models−jump-diffusion processes, i.e., the asset price $\{X(t)\}$ satisfies the following stochastic differential equation (SDE):

$$dX(t) = b(t, X(t))\,dt + \sigma(t, X(t))\,dW(t) + \int_{\mathrm{R}^d} J(X(t), y)\mathcal{P}(dt, dy), \tag{1}$$

where $t \in [0,T]$, $\{X(t) = [X_1(t), ..., X_n(t)]\}$ is a random process with a given initial deterministic value $X(0) = X_0 \in \mathrm{R}^n$, $\{W(t) = [W_1(t), ..., W_{n_w}(t)]\}$ is a standard vector Wiener process, $\mathcal{P}(dt,dy)$ is the Poisson random measure defined on $[0,T] \times \mathrm{R}^d \subset \mathrm{R}^{d+1}$ with the intensity measure $\mu(dt \times dy) = \lambda(dy)dt$ for some finite measure λ on R^d, the coefficients b, σ and J are functions $b : \mathrm{R} \times \mathrm{R}^n \to \mathrm{R}^n$, $\sigma : \mathrm{R} \times \mathrm{R}^n \to \mathrm{R}^n \times \mathrm{R}^{n_w}$ and $J : \mathrm{R}^n \times \mathrm{R}^d \to \mathrm{R}^n$ satisfying mild continuity conditions (such as uniformly Lipschitz continuous or Hölder continuous). Throughout the paper $\mathcal{F} = \{\mathcal{F}_t : 0 \le t \le T\}$ is the augmented filtration generated by the Wiener process $\{W(t)\}$ and the Poisson random measure $\mathcal{P}$.

We consider a Bermudan option based on $\{X(t)\}$ that can be exercised at any date from the time set $\Xi = \{T_0, T_1, ..., T_{\mathcal{J}}\}$, with $T_0 = 0$ and $T_{\mathcal{J}} = T$. Given a pricing measure Q and the filtration $\mathcal{F}$, when exercising at time $T_j \in \Xi$, the holder of the option will receive a discounted payoff:

$$H_{T_j} := h(T_j, X(T_j)), \tag{2}$$

where $h(T_j, \cdot)$ is a Lipschitz continuous function. Our problem is to evaluate the price of the Bermudan option, that is, to find

$$Primal: \qquad V_0^* = \sup_{\tau \in \Xi} E\left[h(\tau, X(\tau))\,|X(0) = X_0\right], \tag{3}$$

where τ is an exercise strategy (in this case, a stopping time adapted to the filtration $\{\mathcal{F}_{T_j} : j = 0, ..., \mathcal{J}\}$) taking values in Ξ, the expectation is taken under the pricing measure Q, and V_0^*

denotes the Bermudan option price at time T_0 given the initial asset price X_0.

As we stated in the previous section, Longstaff and Schwartz (2001) manage to construct a suboptimal exercise strategy $\tilde{\tau}$ and generate a lower bound $V_0^{\tilde{\tau}}$ on the exact option price V_0^* via a least-squares regression approach. We omit the details of their approach and focus on the following dual approach.

2.3 *Review of dual approach*

Let $M = \{M_{T_j} : j = 0, ..., \mathcal{J}\}$ with $M_0 = 0$ be a martingale adapted to the filtration $\{\mathcal{F}_{T_j} : j = 0, ..., \mathcal{J}\}$ and $\mathcal{M}$ represents the set of all such martingales. Anderson and Broadie (2004), Haugh and Kogan (2004) show that the dual problem:

$$Dual: \qquad \inf_{M\in\mathcal{M}} \left(E\left[\max_{0\le j\le\mathcal{J}} \left(H_{T_j} - M_{T_j}\right) | X(0) = X_0 \right] \right) \tag{4}$$

yields the exact option price V_0^*. Moreover, if we let M_{T_j} in (4) be the Doob-Meyer martingale part of the discounted Bermudan price process $V_{T_j}^*$, denoted by $M_{T_j}^*$, then the infimum in (4) is achieved. Precisely, we have:

$$V_0^* = \quad E\left[\max_{0\le j\le\mathcal{J}} \left(H_{T_j} - M_{T_j}^*\right) | X(0) = X_0 \right]. \tag{5}$$

In practice, the optimal dual martingale is not available to us. Nevertheless, we can still obtain an upper bound with an arbitrary $M \in \mathcal{M}$ via

$$V_0^{up}(M) = \quad E\left[\max_{0\le j\le\mathcal{J}} \left(H_{T_j} - M_{T_j}\right) | X(0) = X_0 \right]. \tag{6}$$

It is reasonable to expect that, if M_{T_j} is the martingale induced by a good approximation, V_{T_j}, of the option price process $V_{T_j}^*$, then M_{T_j} is close to the optimal dual martingale $M_{T_j}^*$ and the resulting upper bound $V_0^{up}(M)$ should be close to the exact option price V_0^*. Specifically, suppose $V = \{V_{T_j} : j = 0, ..., \mathcal{J}\}$ is some approximation of $V^* = \{V_{T_j}^* : j = 0, ..., \mathcal{J}\}$. Consider the following Doob-Meyer decomposition:

$$V_{T_j} = V_0 + M_{T_j} + U_{T_j}, \quad j = 0, ..., \mathcal{J}, \tag{7}$$

where V_0 is the approximation of the Bermudan option price at time T_0 and U_{T_j} is the residual predictable process. Then we can obtain the martingale component M_{T_j} in principle via the following recursion:

$$\begin{cases} M_0 = 0, \\ M_{T_{j+1}} = M_{T_j} + V_{T_{j+1}} - E_{T_j}\left[V_{T_{j+1}}\right]. \end{cases} \tag{8}$$

where $E_{T_j}[\cdot]$ means the conditional expectation is taken with respect to the filtration $\mathcal{F}_{T_j}$, i.e., $E_{T_j}[\cdot] = E[\cdot|\mathcal{F}_{T_j}]$.

Haugh and Kogan (2004), Anderson and Broadie (2004) both use the above theoretical result as the starting point of their algorithms to the upper bounds. The difference between their approaches lies in the ways of generating dual martingales. Haugh and Kogan (2004) try to first approximate V^* directly by regressing it on certain function basis and then induce the dual martingale by inner simulation, while Anderson and Broadie (2004) try to first approximate the optimal exercise strategy τ^* by a suboptimal exercise strategy $\tilde{\tau}$, then generate the approximation $V^{\tilde{\tau}}$ of the option price and the corresponding dual martingale by inner simulation.

Due to the nested simulation in approximating the conditional expectation in (8), both of their algorithms introduce zero-mean noise terms in the estimators, preserving martingale property and inducing biased-high upper bounds. A possible drawback of these approaches is the huge computational effort caused by the nested simulation. In this paper, by generalizing the true martingale approach developed by Belomestny *et al.* (2009) to jump-diffusion processes via an effective and efficient discretization scheme on the infinite multi-dimensional jump space, we are able to construct a martingale approximation that preserves the martingale property and induces a true upper bound in a non-nested simulation manner. Due to the relief from the nested simulation, we expect the computational effort to be significantly reduced.

3. True martingale approach via non-nested simulation

In this section, we will develop an approach that is fundamentally different from previous approaches by Haugh and Kogan (2004), Anderson and Broadie (2004). By exploiting the special structure of martingales jointly driven by Wiener measure and Poisson random measure, we are able to construct an approximation of M without nested simulation, which preserves the martingale property. The following generalized martingale representation theorem provides an intuitive idea in understanding the unique structure of such martingales.

Theorem 3.1: [**Martingale Representation Theorem**] *Fix* $T > 0$*. Let* $\{W(t) : 0 \leq t \leq T\}$ *be an* n_w*-dimensional Wiener process and* $\mathcal{P}$ *be a Poisson random measure on* $[0, T] \times \mathrm{R}^d$ *with intensity* $\mu(dt \times dy)$*, independent from* $W(t)$*. If* $M = \{M_{T_j} : j = 0, ..., \mathcal{J}\}$ *is a square-integrable (real-valued) martingale adapted to the filtration* $\{\mathcal{F}_{T_j} : j = 0, ..., \mathcal{J}\}$ *with deterministic initial value* $M_0 = 0$*, then there exist a predictable process* $\phi :\ \Omega \times [0, T] \to \mathrm{R}^{n_w}$ *and a predictable random function* $\psi :\ \Omega \times [0, T] \times \mathrm{R}^d \to \mathrm{R}$ *such that*

$$M_{T_j} = \int_0^{T_j} \phi_s dW(s) + \int_0^{T_j} \int_{\mathrm{R}^d} \psi(s, y) \tilde{\mathcal{P}}(ds, dy), \qquad j = 0, ..., \mathcal{J}, \tag{9}$$

where $\tilde{\mathcal{P}}$ *is the compensated Poisson random measure induced by* $\mathcal{P}$*.*

Proof: According to Proposition 9.4 in Cont and Tankov (2003), for the random variable M_T, there exist a predictable process $\phi :\ \Omega \times [0, T] \to \mathrm{R}^{n_w}$ and a predictable random function $\psi :\ \Omega \times [0, T] \times \mathrm{R}^d \to \mathrm{R}$ such that

$$M_T = \int_0^{T} \phi_s dW(s) + \int_0^{T} \int_{\mathrm{R}^d} \psi(s, y) \tilde{\mathcal{P}}(ds, dy),$$

where $\tilde{\mathcal{P}}$ is the compensated Poisson random measure induced by $\mathcal{P}$. Given that M is a martingale adapted to the filtration $\{\mathcal{F}_{T_j} : j = 0, ..., \mathcal{J}\}$, and according to the zero-mean and independent increment properties of Wiener measure W and compensated Poisson random measure $\tilde{\mathcal{P}}$, we have

$$M_{T_j} = E\left[M_T | \mathcal{F}_{T_j}\right] = \int_0^{T_j} \phi_s dW(s) + \int_0^{T_j} \int_{\mathrm{R}^d} \psi(s, y) \tilde{\mathcal{P}}(ds, dy), \quad \text{for} \quad j = 0, ..., \mathcal{J}.$$

$\square$

Theorem 3.1 can be interpreted as a generalization of the martingale representation theorem for martingales driven by Wiener processes. Indeed, if the intensity $\mu(dt \times dy)$ equals zero, then Theorem 3.1 reduces to the classic martingale representation theorem. Moreover, similar to the Wiener measure W, the compensated Poisson random measure $\tilde{\mathcal{P}}$ possesses the zero-mean and independent increment properties, which are essential for the true martingale approximation to preserve the martingale property. However, Theorem 3.1 fails to provide an intuitive understanding towards the explicit relationship between ϕ_t, $\psi(t, y)$ and the martingale M itself. In the following theorem, we complement this deficiency by explicitly expressing the integrands as functions of the process that induces the martingale.

Theorem 3.2 : *Suppose X_t follows (1) and induces the jump measure $\mathcal{P}_X$, which is a Poisson random measure. Consider the American option price process (continuous-time) $(V_t)_{0\leq t\leq T}$ with payoff of the form $h(\cdot,\cdot)$ in (2). Assuming $V_t = v^a(t, X(t))$ is a C^2 function in X and its two partial derivatives are bounded by a constant, then the martingale component of $(V_t - V_0)_{0\leq t\leq T}$, denoted by $(M_t)_{0\leq t\leq T}$ with $M_0 = 0$, is given by:*

$$M_t = \int_0^t \frac{\partial v^a\left(s^-, X_{s^-}\right)}{\partial X} \sigma dW_s + \int_0^t \int_{\mathrm{R}^d} \left[v^a\left(s^-, X_{s^-} + y\right) - v^a\left(s^-, X_{s^-}\right)\right] \tilde{\mathcal{P}}_X\left(ds, dy\right), \quad 0 \leq t \leq T, \tag{10}$$

where $\tilde{\mathcal{P}}_X\left(ds, dy\right)$ is the compensated Poisson random measure induced by $\mathcal{P}_X$.

Proof: Apply Proposition 8.16 in Cont and Tankov (2003) to jump process X_t and we can immediately obtain the result above. □

Remark 1 :

(i) In a continuous-time setting, Theorem 3.2 reveals the structure of the optimal dual martingale. Furthermore, one may directly construct an approximation of it by simply replacing the exact American option process with an approximate American option process in (10).

(ii) In a discrete-time setting, Theorem 3.2 implies that ϕ_t is close to the derivative of the Bermudan option price, while $\psi(t, y)$ is close to the Bermudan option price increment caused by the jump. If we want to approximate the integrands ϕ_t and $\psi(t, y)$, we should start with the derivative and the increment of certain option price which is close to Bermudan option price and admits a closed-form, e.g. the European option price. Specifically, in the subsequent section, when constructing the true martingale approximation, we need to use least-squares regression method to approximate the integrands. Therefore, we should incorporate the derivative and the increment of the European option price into the function bases for regressing ϕ_t and $\psi(t, y)$, respectively.

(iii) In practice, the asset price process X_t, which is usually an exponential of a compound Poisson process (see the numerical example (29) in Section 5), induces a complicated jump measure P_X that can hardly be simulated. To apply Theorem 3.2, we can introduce an intermediate function $S(\cdot)$ such that $S(t) = S(X(t))$ induces a relatively easy-to-simulate Poisson random measure P_S without significantly increasing the complexity of the function $v^a(\cdot,\cdot)$. In other words, the choice of the Poisson random measure $\mathcal{P}_S$ (or function $S(\cdot)$) is essential to simplify the representation of the martingale. In fact, the choice of $S(\cdot)$ should balance the tradeoff between the complexity of $\mathcal{P}_S$ and the complexity of function $v(\cdot,\cdot)$. For example, for the asset price process X_t satisfying (29), a good choice for the intermediate function $S(\cdot)$ is the natural logarithm function $\ln(\cdot)$. More details of this choice will be presented in Section 5.

Inspired by Theorem 3.1 and Theorem 3.2, as well as Belomestny *et al.* (2009)'s idea of "true martingale", if one tries to approximate the martingale M_{T_j}, a natural idea is to first estimate the integrands ϕ_t and $\psi\left(t, y\right)$ in the expression

$$M_{T_j} = \int_0^{T_j} \phi_t dW\left(t\right) + \int_0^{T_j} \int_{\mathrm{R}^d} \psi\left(t, y\right) \tilde{\mathcal{P}}\left(dt, dy\right), \quad j = 0, ..., \mathcal{J}, \tag{11}$$

at a finite number of time and space points. Then an approximation of M_{T_j} will be represented via ϕ_t and $\psi\left(t, y\right)$ using the Ito sum (similar to the Riemann sum).

We introduce a partition $\pi = \{t_l : l = 0, 1, ..., \mathcal{L}\}$ on $[0, T]$ such that $t_0 = 0$, $t_{\mathcal{L}} = T$ and

$\pi \supset \Xi$, and a partition $\mathcal{A} = \{A_k : k = 0, 1, ..., \mathcal{K}\}$ on R^d such that A_k are λ-measurable disjoint subsets and $\bigcup_{k=1}^{\mathcal{K}} A_k = \mathrm{R}^d$, where we recall $\mu(dt \times dy) = \lambda(dy)dt$ for some finite measure λ on R^d. Therefore, $P([t_l, t_{l+1}] \times A_k) := \int_{t_l}^{t_{l+1}} \int_{A_k} \mathcal{P}(ds, dy)$ is a Poisson random variable (regarded as Poisson increment), and $\tilde{P}([t_l, t_{l+1}] \times A_k) := \int_{t_l}^{t_{l+1}} \int_{A_k} \tilde{\mathcal{P}}(ds, dy)$ is the corresponding compensated Poisson random variable (regarded as compensated Poisson increment) with intensity $\mu([t_l, t_{l+1}] \times A_k) := \int_{t_l}^{t_{l+1}} \int_{A_k} \lambda(dy)dt$. We further denote the magnitude of partitions π and $\mathcal{A}$ as $|\pi|$ and $|\mathcal{A}|$ respectively, i.e., $|\pi| = \max_{0<l\leq\mathcal{L}} (t_l - t_{l-1})$, $|A_k| = \int_{A_k} \lambda(dy)$, and $|\mathcal{A}| = \max_{1\leq k\leq\mathcal{K}} |A_k|$.

From (7), we have

$$V_{T_{j+1}} - V_{T_j} = \left(M_{T_{j+1}} - M_{T_j}\right) + \left(U_{T_{j+1}} - U_{T_j}\right), \quad j = 0, ..., \mathcal{J}. \tag{12}$$

Combining with the Ito sum of $M_{T_{j+1}}$ in (11), we have

$$\begin{aligned} V_{T_{j+1}} - V_{T_j} \approx & \sum_{T_j \leq t_l < T_{j+1}} \phi_{t_l} \left(W(t_{l+1}) - W(t_l)\right) \\ & + \sum_{T_j \leq t_l < T_{j+1}} \sum_{k=1}^{\mathcal{K}} \psi(t_l, y_k) \tilde{P}([t_l, t_{l+1}] \times A_k) + U_{T_{j+1}} - U_{T_j}, \end{aligned} \tag{13}$$

where $y_k \in A_k$ is a representative value, and we will keep using this notation thereafter. Multiplying both sides of (13) by the Wiener process increment $(W(t_{l+1}) - W(t_l))$ and taking conditional expectations with respect to the filtration $\mathcal{F}_{t_l}$, we obtain

$$\phi_{t_l} \approx \frac{1}{t_{l+1} - t_l} E_{t_l} \left[\left(W(t_{l+1}) - W(t_l)\right) V_{T_{j+1}}\right], \quad T_j \leq t_l < T_{j+1}, \tag{14}$$

where we use the $\mathcal{F}$-predictability of U, the independent increment property of $W(t)$ and the independence between W and $\mathcal{P}$.

Similarly, if we multiply both sides of (13) by the compensated Poisson random variable $\tilde{P}([t_l, t_{l+1}] \times A_k)$ and take the conditional expectations with respect to the filtration $\mathcal{F}_{t_l}$, we can obtain

$$\psi(t_l, y_k) \approx \frac{1}{\mu([t_l, t_{l+1}] \times A_k)} E_{t_l} \left[\tilde{P}([t_l, t_{l+1}] \times A_k) V_{T_{j+1}}\right], T_j \leq t_l < T_{j+1}, 1 \leq k \leq \mathcal{K}, \tag{15}$$

where we again use the $\mathcal{F}$-predictability of U, the independent increment property of $\mathcal{P}$ and the independence between $\mathcal{P}$ and W.

Motivated by expressions (14) and (15), we denote the approximation of ϕ_{t_l} and $\psi(t_l, y_k)$ by $\phi^{\pi}_{t_l}$ and $\psi^{\pi,\mathcal{A}}(t_l, y_k)$ respectively, which are defined as follows:

$$\phi^{\pi}_{t_l} := \frac{1}{\Delta^{\pi}_l} E_{t_l} \left[\left(\Delta^{\pi} W_l\right) V_{T_{j+1}}\right], \quad T_j \leq t_l < T_{j+1}, \tag{16}$$

and

$$\psi^{\pi,\mathcal{A}}(t_l, y_k) := \frac{1}{\mu([t_l, t_{l+1}] \times A_k)} E_{t_l} \left[\tilde{P}([t_l, t_{l+1}] \times A_k) V_{T_{j+1}}\right], T_j \leq t_l < T_{j+1}, 1 \leq k \leq \mathcal{K}, \tag{17}$$

where Δ^{π}_l and $\Delta^{\pi} W_l$ represent the increments of time t and the Winer process $W(t)$ respectively, i.e. $\Delta^{\pi}_l = (t_{l+1} - t_l)$ and $\Delta^{\pi} W_l = (W_{l+1} - W_l)$. Therefore, we can construct an approximation of M_{T_j}, denoted by $M_{T_j}^{\pi,\mathcal{A}}$, as follows:

$$M_{T_j}^{\pi,\mathcal{A}} := \sum_{0 \le t_l < T_j} \phi_{t_l}^{\pi} \cdot (\Delta^{\pi} W_l) + \sum_{0 \le t_l < T_j} \sum_{k=1}^{\mathcal{K}} \psi^{\pi,\mathcal{A}}(t_l, y_k)\, \tilde{P}([t_l, t_{l+1}] \times A_k). \tag{18}$$

The construction procedure of $M_{T_j}^{\pi,\mathcal{A}}$ can be summarized in the following algorithm.

Algorithm 1: ***Construction of the Martingale Approximation*** $M^{\pi,\mathcal{A}}$

- Step 1: Express M_{T_j} as an integral of $\phi(t)$ and $\psi(t,y)$ via (11).
- Step 2: Approximate ϕ_{t_l} by $\phi_{t_l}^{\pi}$ via (16) and $\psi(t_l, y_k)$ by $\psi^{\pi,\mathcal{A}}(t_l, y_k)$ via (17) respectively.
- Step 3: Construct the approximation of M_{T_j}, denoted by $M_{T_j}^{\pi,\mathcal{A}}$, via (18).

Under the pure-diffusion models, Belomestny *et al.* (2009) construct the approximation of M_{T_j}, denoted by $M_{T_j}^{\pi}$, to preserve the martingale property. Here we have generalized their techniques to the approximation of the jump part of the martingale under the jump-diffusion models. We observe that, similar to regarding ϕ_t as a random function of time, we can regard $\psi(t,y)$ in (13) as a random function of both time and space variables. By properly constructing the Poisson random measure and partitioning the supporting space R^d with respect to (w.r.t.) the Poisson random measure, we are able to construct the Ito sum to approximate the jump part of M_{T_j}.

Notice that $M^{\pi,\mathcal{A}} = \{M_{T_j}^{\pi,\mathcal{A}} : j = 0, ..., \mathcal{J}\}$ remains to be a martingale adapted to the filtration $\{\mathcal{F}_{T_j} : j = 0, ..., \mathcal{J}\}$, based on its structure. We formally state this result in the following theorem.

Theorem 3.3: [**True Martingale**] *If an approximation of M, denoted by $M^{\pi,\mathcal{A}}$, is constructed using Algorithm* **1**, *then $M^{\pi,\mathcal{A}}$ is still a martingale adapted to the filtration $\{\mathcal{F}_{T_j} : j = 0, ..., \mathcal{J}\}$.*

Proof: To show $M^{\pi,\mathcal{A}}$ is a martingale adapted to the filtration $\{\mathcal{F}_{T_j} : j = 0, ..., \mathcal{J}\}$, it is sufficient to show that for $0 \le j_1 < j_2 \le \mathcal{J}$, $E\left[M_{T_{j_2}} | \mathcal{F}_{T_{j_1}}\right] = M_{T_{j_1}}$.

For $0 \le l \le \mathcal{L}$ and $1 \le k \le \mathcal{K}$, $\phi_{t_l}^{\pi}$ and $\psi^{\pi,\mathcal{A}}(t_l, y_k)$ are functions of t_l and X_{t_l}. Hence, they are independent from both $\Delta^{\pi} W_l$ and $\tilde{P}([t_l, t_{l+1}] \times A_k)$. According to the zero-mean property of $\Delta^{\pi} W_l$ and $\tilde{P}([t_l, t_{l+1}] \times A_k)$, we have

$$\begin{aligned} E\left[M_{T_{j_2}}^{\pi,\mathcal{A}} | \mathcal{F}_{T_{j_1}}\right] &= E\left[\sum_{0 \le t_l < T_{j_2}} \phi_{t_l}^{\pi} \cdot (\Delta^{\pi} W_l) + \sum_{0 \le t_l < T_{j_2}} \sum_{k=1}^{\mathcal{K}} \psi^{\pi,\mathcal{A}}(t_l, y_k)\, \tilde{P}([t_l, t_{l+1}] \times A_k) \Big| \mathcal{F}_{T_{j_1}}\right] \\ &= \sum_{0 \le t_l < T_{j_1}} \phi_{t_l}^{\pi} \cdot (\Delta^{\pi} W_l) + \sum_{0 \le t_l < T_{j_1}} \sum_{k=1}^{\mathcal{K}} \psi^{\pi,\mathcal{A}}(t_l, y_k)\, \tilde{P}([t_l, t_{l+1}] \times A_k) \\ &= M_{T_{j_1}}^{\pi,\mathcal{A}}. \end{aligned}$$

□

Because of Theorem 3.3, we refer $M^{\pi,\mathcal{A}}$ as the "true martingale approximation". Furthermore, if we plug $M^{\pi,\mathcal{A}}$ in (6), it is easy to see that $V_0^{up}\left(M^{\pi,\mathcal{A}}\right)$ is a "true upper bound" on the Bermudan option price V_0^*.

3.1 *Error analysis*

A natural question that arises after we obtain $M^{\pi,\mathcal{A}}$ is how good it approximates the objective martingale M. Next, we will focus on the limiting behaviour of $M^{\pi,\mathcal{A}}$ w.r.t. the partitions on time and jump spaces, and bound the distance between $M^{\pi,\mathcal{A}}$ and M with $\mathcal{L}$ (# of time partitioning points) and $\mathcal{K}$ (# of space partitioning sets).

Consider the following specific partitioning procedure. For the partition π on the time space, we simply perform an equi-length partition. For the partition $\mathcal{A}$ on the jump space R^d, we let $A_{\mathcal{K}} := \{y \in \mathrm{R}^d : \|y\| > \frac{K}{2}\}$, where $\|.\|$ denotes the Euclidean norm, and then perform a $\overbrace{K^2 \times K^2 \times \cdots \times K^2}^{d}$ equi-distance grid partition on $\mathrm{R}^d \setminus A_{\mathcal{K}} = \{y \in \mathrm{R}^d : \|y\| \le \frac{K}{2}\}$ to obtain K^{2d} partitioning sets $\{A_k : 1 \le k \le K^{2d}\}$. Note that $\|y - \tilde{y}\|_2 \le \frac{1}{K}, \forall y, \tilde{y} \in A_k \; \forall k = 1, ..., K^{2d}$. In total, there are $\mathcal{K} = K^{2d} + 1$ partitioning subsets. Under this specific partitioning procedure, we have the following theorem.

Theorem 3.4: *Let M_{T_j} be the martingale component of $V_{T_j} = v\left(T_j, X_{T_j}\right)$ and $M^{\pi,\mathcal{A}}_{T_j}$ be its approximation obtained via Algorithm **1**. Assume the measure λ on jump space R^d has finite second moment, i.e., $\int_{\mathrm{R}^d} \|y\|^2 \lambda(dy) < \infty$. Then there exist constants $C > 0$ and $C' > 0$ such that, $\forall \mathcal{L}$ and $\forall \mathcal{K} = K^{2d} + 1$,*

$$E\left[\max_{0 \le j \le \mathcal{J}} |M^{\pi,\mathcal{A}}_{T_j} - M_{T_j}|^2\right] \le C \cdot \frac{1}{\mathcal{L}} + C' \cdot \frac{1}{K^2} + \delta(K), \tag{19}$$

where $\delta(K) \to 0$ as $K \to \infty$. Furthermore, if the tail of λ's second moment, i.e. $\int_{\|y\| > \frac{K}{2}} \|y\|^2 \lambda(dy)$, decays "at least" quadratically in K, then (19) can be further strengthened to

$$E\left[\max_{0 \le j \le \mathcal{J}} |M^{\pi,\mathcal{A}}_{T_j} - M_{T_j}|^2\right] \le C \cdot \frac{1}{\mathcal{L}} + \hat{C} \cdot \frac{1}{K^2} \tag{20}$$

for some constant $\hat{C}$.

Proof: See Appendix A. □

According to the relationship between M and $V_0^{up}(M)$ in (6), we can immediately obtain the following corollary on the quality of upper bounds.

Corollary 3.5: *Under the assumptions of Theorem 3.4, we have*

$$|V_0^{up}(M^{\pi,\mathcal{A}}) - V_0^{up}(M)|^2 \le C \cdot \frac{1}{\mathcal{L}} + \hat{C} \cdot \frac{1}{K^2}.$$

Corollary 3.5 implies that sufficient fine partitions π and $\mathcal{A}$ lead to a good estimator of the upper bound $V_0^{up}(M)$ on the Bermudan option price.

Remark 2:

(i) The proof of Theorem 3.4 boils down to characterizing the path regularity properties of the integrands $\phi(s)$ and $\psi(s, y)$ in the martingale representation. Thanks to the beautiful analysis in Bouchard and Elie (2008), we are able to show the results.

(ii) The assumption $\int_{\mathrm{R}^d} \|y\|^2 \lambda(dy) < \infty$ on the finite measure λ is quite realistic. For λ induced by the most commonly used jump processes in financial modelling, i.e., jump processes induced by jump distributions with exponentially decaying densities, the finite second moment assumption of measure λ holds.

(iii) Notice that, in order to maintain the same level of theoretical error bound, $\mathcal{K}$—the # of partitioning sets are drastically increasing as the dimension of jump space increases, which is undesirable from the perspective of practical implementation. Fortunately, in practice, we can take advantage of the structure of λ to construct a very efficient partition, as an alternative to the partitioning procedure suggested above. When the measure λ is induced by a jump distribution with exponentially decaying tail probability (see Proposition 5.1), which is the case for most of jump distributions used in financial modelling, a very coarse partition $\mathcal{A}$ or a small $\mathcal{K}$ is sufficiently good if one considers a equi-probability partition with connected subsets. The reason is that most of the jump realizations will fall into a small neighborhood around the mean of the jump distribution, and we only need to use one partitioning set to cover the area outside of the neighborhood, and then focus on finer partitioning of the small neighborhood around the mean. For example, considering the case of a standard normal distribution with mean 0 and variance 1, the jump size takes value in $[-2,2]$ with probability approximately 95%. Therefore, a small number of partitioning sets for $[-2,2]$ are sufficient to capture most of the jump realizations. In fact, our numerical results show that, even with $\mathcal{K}=10$, the martingale approximation generates tight upper bounds on the Bermudan option price. We will discuss more details in the numerical results in Section 5.

In Theorem 3.4, the extra assumption that the tail of λ's second moment decays "at least" quadratically in K makes it possible to eliminate the term $\delta(K)$ in (19) and obtain a "cleaner" error bound in (20). From (A10) in Appendix A, we know that $\delta(K)$ comes from the discretization error accumulated in $A_{\mathcal{K}}=\{y\in \mathrm{R}^d : \|y\|>\frac{K}{2}\}$, and it satisfies

$$\delta(K)=4T\tilde{C}'\left(\int_{\|y\|>\frac{K}{2}}\|y\|^2\lambda(dy)-\frac{\left(\int_{\|y\|>\frac{K}{2}}\|y\|\lambda(dy)\right)^2}{\int_{\|y\|>\frac{K}{2}}\lambda(dy)}\right)\leq 4T\tilde{C}'\left(\int_{\|y\|>\frac{K}{2}}\|y\|^2\lambda(dy)\right), \qquad (21)$$

!!where T is the time horizon and $\tilde{C}'$ is a constant. If $\int_{\|y\|>\frac{K}{2}}\|y\|^2\lambda(dy)$—the tail of λ's second moment decays "at least" quadratically in K, then $\delta(K)$ decays "at least" quadratically in K and collapses into the term $C'\cdot\frac{1}{K^2}$. Furthermore, notice that (21) only involves the measure, the first and second moment of λ, we can easily quantify the decay rate of $\delta(K)$ and verify the assumptions in Theorem 3.4 for λ induced by some specific jump distributions. In particular, in the following proposition we quantify the decay rate of $\delta(K)$ for two specific families of jump distributions, in coverage of the most commonly used jump processes in financial modelling.

Proposition 3.6: *(i) Consider the family of jump distributions with polynomially decaying densities, i.e.,* $\lambda(dy)\propto\frac{1}{\|y\|^\zeta}dy$, *where* $\zeta>d+2$ *and* d *is the dimension of the jump space, then* $\delta(K)\propto\frac{1}{K^{\zeta-d-2}}$. *Furthermore, if* $\zeta>d+4$, *then (20) holds.*
(ii) Consider the family of jump distributions with exponentially decaying densities, i.e., $\lambda(dy)\propto\exp(-\|y\|^\eta)dy$, *where* $\eta>0$. *Then,* $\delta(K)\propto\mathcal{C}(K)\exp(-K^\eta)$, *where* $\mathcal{C}(\cdot)$ *is bounded polynomially. Furthermore, (20) holds.*

Proof: The results are immediate by substituting the explicit form of $\lambda(dy)$ into (21) and applying multivariate calculus with basic change of variables. □

Proposition 3.6 indicates that, at least for the jump processes induced by jump distributions with exponentially decaying densities, e.g., the norm distribution proposed in Merton (1976) and the family of exponential distributions proposed in Kou (2002), Cai and Kou (2011), and Cai and Kou (2012), our proposed partition technique on the jump space guarantees an intuitive bound on the discretization error.

4. True martingale algorithm

In theory, we show that the true martingale approximation incurs bounded error. Now let us focus on the practical implementation. Notice that using Monte Carlo simulation to directly estimate the conditional expectations in (16) and (17) incurs nested simulation. Therefore, this approach is highly undesirable from the perspective of computational efficiency. To improve the computational efficiency, we need to further manipulate (16) and (17), and then seek an alternative approach to estimating ϕ^π and $\psi^{\pi,\mathcal{A}}$ in hope of avoiding nested simulation. The idea comes from the least-squares regression method applied in Longstaff and Schwartz (2001).

Suppose the approximate value function V in (16) and (17) is induced by a suboptimal exercising strategy (stopping time) $\tilde{\tau}$ via $V_{T_j} = E_{T_j}[H_{\tilde{\tau}_j}]$, where $\tilde{\tau}_j$ means the stopping time $\tilde{\tau}$ takes value greater than or equal to j. Due to the tower property of conditional expectations, we can rewrite (16) and (17) as

$$\phi^\pi_{t_l} := \frac{1}{\Delta^\pi_l} E_{t_l}\left[(\Delta^\pi W_l) H_{\tilde{\tau}_{j+1}}\right], \quad T_j \le t_l < T_{j+1}, \tag{22}$$

and

$$\psi^{\pi,\mathcal{A}}(t_l, y_k) := \frac{1}{\mu([t_l, t_{l+1}] \times A_k)} E_{t_l}\left[\tilde{P}([t_l, t_{l+1}] \times A_k) H_{\tilde{\tau}_{j+1}}\right], T_j \le t_l < T_{j+1}, 1 \le k \le \mathcal{K}. \tag{23}$$

Now the approximate integrands ϕ^π and $\psi^{\pi,\mathcal{A}}$ are expressed as functions of the suboptimal policy $\tilde{\tau}$, instead of the approximate Bermudan option price V. Therefore, we can apply the least-squares regression method to estimate the approximate integrands ϕ^π and $\psi^{\pi,\mathcal{A}}$ without involving nested simulation, similar to the regression method applied in Longstaff and Schwartz (2001).

Now we can formally describe the T-M algorithm based on the construction of the martingale approximation $M^{\pi,\mathcal{A}}$ in Section 3. The outline of the T-M algorithm consists of four steps in order: generating a suboptimal exercise strategy $\tilde{\tau}$, approximating the integrands ϕ^π and $\psi^{\pi,\mathcal{A}}$, constructing the martingale approximation $M^{\pi,\mathcal{A}}$, and generating true upper bounds $V_0^{up}\left(\hat{M}^{\pi,\mathcal{A}}\right)$ on the Bermudan option price.

First, let us start with generating the suboptimal exercise strategy $\tilde{\tau}$. It not only provides the lower bound, but also plays an important role in approximating the integrands ϕ^π and $\psi^{\pi,\mathcal{A}}$ due to the use of regression. We adopt the L-S algorithm to generate the suboptimal exercise strategy $\tilde{\tau}$ and the corresponding approximate Bermudan option process $\bar{V}_{T_j}$, of the form

$$\bar{V}_{T_j} = E_{T_j}\left[H_{\tilde{\tau}_j}\left(X^{\bar{\pi},\bar{\mathcal{A}}}\right)\right], \tag{24}$$

where $\bar{\pi}(\supset \pi)$, $\bar{\mathcal{A}}(\supset \mathcal{A})$ are employed to simulate the discretized asset price process $\{X^{\bar{\pi},\bar{\mathcal{A}}}\}$.

Second, let us approximate the integrands ϕ^π and $\psi^{\pi,\mathcal{A}}$. To avoid confusion, we denote

$$\begin{cases} \bar{\phi}^\pi_{t_l} = \frac{1}{\Delta^\pi_l} E_{t_l}\left[(\Delta^\pi W_l) H_{\tilde{\tau}_{j+1}}\left(X^{\bar{\pi},\bar{\mathcal{A}}}\right)\right], \quad T_j \le t_l < T_{j+1} \\ \bar{\psi}^{\pi,\mathcal{A}}(t_l, y_k) = \frac{1}{\mu([t_l,t_{l+1}]\times A_k)} E_{t_l}\left[\tilde{P}([t_l, t_{l+1}] \times A_k) H_{\tilde{\tau}_{j+1}}\left(X^{\bar{\pi},\bar{\mathcal{A}}}\right)\right], T_j \le t_l < T_{j+1}, 1 \le k \le \mathcal{K} \end{cases}$$

as the counterparts of $\phi^\pi_{t_l}$ and $\psi^{\pi,\mathcal{A}}(t_l, y_k)$ respectively, under the discretized asset price $X^{\bar{\pi},\bar{\mathcal{A}}}$. Recall that we need to approximate $\bar{\phi}^\pi_{t_l}$ and $\bar{\psi}^{\pi,\mathcal{A}}(t_l, y_k)$ via regression, to achieve the goal of non-nested simulation. In particular, note that $\bar{\phi}^\pi_{t_l}$ and $\bar{\psi}^{\pi,\mathcal{A}}(t_l, y_k)$ both are implicit functions of the underlying asset price process $X^{\bar{\pi},\bar{\mathcal{A}}}_{t_l}$ (see the conditional expectations w.r.t. $\mathcal{F}_{t_l}$ in (22) and (23)), we choose the function bases to be row function vectors $\rho^W\left(t_l, X^{\bar{\pi},\bar{\mathcal{A}}}_{t_l}\right) = \left(\rho^W_i(t_l, X^{\bar{\pi},\bar{\mathcal{A}}}_{t_l})\right)_{i=1,\dots,I_1}$ and $\rho^{\mathcal{P}}\left(t_l, y_k, X^{\bar{\pi},\bar{\mathcal{A}}}_{t_l}\right) = \left(\rho^{\mathcal{P}}_i(t_l, y_k, X^{\bar{\pi},\bar{\mathcal{A}}}_{t_l})\right)_{i=1,\dots,I_2}$ respectively, where I_1 and I_2 are the dimensions of the function bases. If we simulate N independent samples of Wiener increments $\Delta^\pi W_l$, denoted by $\{\Delta^\pi_n W_l : \ l = 1, \dots\mathcal{L}, n = 1, \dots, N\}$, and N independent samples of Poisson increments $P([t_l, t_{l+1}] \times A_k)$, denoted by $\{P_n([t_l, t_{l+1}] \times A_k) : l = 1, \dots, \mathcal{L}, k = 1, \dots, \mathcal{K}, n = 1, \dots, N\}$, and based on those samples we construct the sample paths of the asset price $\{X^{\bar{\pi},\bar{\mathcal{A}}}_{t_l,n}\}_{l=0,\dots,\mathcal{L},n=1,\dots,N}$, then we can obtain the regressed coefficients $\hat{\alpha}_{t_l}$ and $\hat{\beta}_{t_l,k}$ for $T_j \leq t_l < T_{j+1}$ and $1 \leq k \leq \mathcal{K}$, via

$$\begin{cases} \hat{\alpha}_{t_l} & = \arg \min\limits_{\alpha \in \mathrm{R}^{I_1}} \left\{ \sum\limits_{n=1}^{N} \left| \frac{\Delta_n^\pi W_l}{\Delta_l^\pi} H_{\tilde{\tau}_{j+1}} \left(X_{\cdot,n}^{\bar{\pi},\bar{\mathcal{A}}} \right) - \rho^W \left(t_l, X_{t_l,n}^{\bar{\pi},\bar{\mathcal{A}}} \right) \alpha \right|^2 \right\} \\ \hat{\beta}_{t_l,k} & = \arg \min\limits_{\beta \in \mathrm{R}^{I_2}} \left\{ \sum\limits_{n=1}^{N} \left| \frac{\tilde{P}_n([t_l,t_{l+1}] \times A_k)}{\mu([t_l,t_{l+1}] \times A_k)} H_{\tilde{\tau}_{j+1}} \left(X_{\cdot,n}^{\bar{\pi},\bar{\mathcal{A}}} \right) - \rho^{\mathcal{P}} \left(t_l, y_k, X_{t_l,n}^{\bar{\pi},\bar{\mathcal{A}}} \right) \beta \right|^2 \right\} \end{cases}, \tag{25}$$

where $X_{\cdot,n}^{\bar{\pi},\bar{\mathcal{A}}}$ denotes the nth sample path of the discretized asset price $X^{\bar{\pi},\bar{\mathcal{A}}}$, i.e., $X_{\cdot,n}^{\bar{\pi},\bar{\mathcal{A}}} = \{X_{t_l,n}^{\bar{\pi},\bar{\mathcal{A}}}\}_{l=0,...,\mathcal{L}}$. Now we can compute the estimates of the integrands $\bar{\phi}_{t_l}^\pi$ and $\bar{\psi}^{\pi,\mathcal{A}}(t_l, y_k)$, denoted by $\hat{\phi}^\pi \left(t_l, X_{t_l}^{\bar{\pi},\bar{\mathcal{A}}} \right)$ and $\hat{\psi}^{\pi,\mathcal{A}} \left(t_l, y_k, X_{t_l}^{\bar{\pi},\bar{\mathcal{A}}} \right)$ respectively, via

$$\begin{cases} \hat{\phi}^\pi \left(t_l, X_{t_l}^{\bar{\pi},\bar{\mathcal{A}}} \right) = \rho^W \left(t_l, X_{t_l}^{\bar{\pi},\bar{\mathcal{A}}} \right) \hat{\alpha}_{t_l} \\ \hat{\psi}^{\pi,\mathcal{A}} \left(t_l, y_k, X_{t_l}^{\bar{\pi},\bar{\mathcal{A}}} \right) = \rho^{\mathcal{P}} \left(t_l, y_k, X_{t_l}^{\bar{\pi},\bar{\mathcal{A}}} \right) \hat{\beta}_{t_l,k} \end{cases}. \tag{26}$$

Next, we construct an approximation of $M^{\pi,\mathcal{A}}$, denoted by $\hat{M}^{\pi,\mathcal{A}}$, by replacing the approximate integrands ϕ^π and $\psi^{\pi,\mathcal{A}}$ in the Ito sum (18) with the regressed integrands $\hat{\phi}^\pi$ and $\hat{\psi}^{\pi,\mathcal{A}}$. Precisely, we have

$$\hat{M}_{T_j}^{\pi,\mathcal{A}} := \sum_{0 \leq t_l < T_j} \hat{\phi}^\pi \left(t_l, X_{t_l}^{\bar{\pi},\bar{\mathcal{A}}} \right) \cdot (\Delta^\pi W_l) + \sum_{0 \leq t_l < T_j} \sum_{k=1}^{\mathcal{K}} \hat{\psi}^{\pi,\mathcal{A}} \left(t_l, y_k, X_{t_l}^{\bar{\pi},\bar{\mathcal{A}}} \right) \tilde{P} \left([t_l, t_{l+1}] \times A_k \right). \tag{27}$$

It is necessary to point out that $\hat{M}^{\pi,\mathcal{A}}$ remains to be a martingale adapted to the filtration $\{\mathcal{F}_{T_j} : j = 0, ..., \mathcal{J}\}$, thus $V_0^{up} \left(\hat{M}^{\pi,\mathcal{A}} \right)$ remains to be a true upper bound on the Bermudan option price V_0^*.

Finally, let us estimate $V_0^{up} \left(\hat{M}^{\pi,\mathcal{A}} \right)$ via Monte Carlo simulation by simulating a new set of $\bar{N}$ independent sample paths $\{X_{\cdot,n}^{\bar{\pi},\bar{\mathcal{A}}} : n = 1, ..., \bar{N}\}$ and plugging the realization of $\hat{M}^{\pi,\mathcal{A}}$ along the sample path $X_{\cdot,n}^{\bar{\pi},\bar{\mathcal{A}}}$, denoted by $\hat{M}_{\cdot,n}^{\pi,\mathcal{A}}$, into the dual formulation (6). Precisely, an unbiased estimator for $V_0^{up} \left(\hat{M}^{\pi,\mathcal{A}} \right)$ is given as follows:

$$\hat{V}_0^{up} \left(\hat{M}^{\pi,\mathcal{A}} \right) = \frac{1}{\bar{N}} \sum_{n=1}^{\bar{N}} \max_{0 \leq j \leq \mathcal{J}} \left[h \left(T_j, X_{T_j,n}^{\bar{\pi},\bar{\mathcal{A}}} \right) - \hat{M}_{T_j,n}^{\pi,\mathcal{A}} \right]. \tag{28}$$

We formally summarize these steps in the following algorithm.

Algorithm 2: ***True Martingale Algorithm***

- Step 1: Apply the L-S algorithm to generate a suboptimal exercise strategy $\tilde{\tau}$.
- Step 2: Simulate N independent samples of Wiener increments $\Delta^\pi W_l$ and N independent samples of Poisson increments $P\left([t_l, t_{l+1}] \times A_k\right)$, for $l = 0, ..., \mathcal{L} - 1$ and $k = 1, ..., \mathcal{K}$; construct the sample paths of the asset price $\{X_{\cdot,n}^{\bar{\pi},\bar{\mathcal{A}}} : n = 1, ..., N\}$.
- Step 3: Obtain the parameters $\hat{\alpha} = \{\hat{\alpha}_{t_l}\}_{l=0,...,\mathcal{L}}$ and $\hat{\beta} = \{\hat{\beta}(t_l, k)\}_{l=0,...,\mathcal{L},k=0,...,\mathcal{K}}$ via least-squares regression (25) with exercising $\tilde{\tau}$ along the sample paths $\{X_{\cdot,n}^{\bar{\pi},\bar{\mathcal{A}}} : n = 1, ..., N\}$.
- Step 4: Simulate a new set of $\bar{N}$ independent sample paths $\{X_{\cdot,n}^{\bar{\pi},\bar{\mathcal{A}}} : n = 1, ..., \bar{N}\}$; compute $\hat{\phi}^\pi$ and $\hat{\psi}^{\pi,\mathcal{A}}$ via (26); construct the martingale approximation $\hat{M}^{\pi,\mathcal{A}}$ via (27); obtain an unbiased estimator $\hat{V}_0^{up}(\hat{M}^{\pi,\mathcal{A}})$ for the true upper bound on the Bermudan option price V_0^* via (28).

In the remainder of this section, we give a bound on the error of $\hat{M}^{\pi,\mathcal{A}}$, similar to the bound in Theorem (3.4), in approximating the objective martingale M_{T_j} provided that the errors incurred by the Euler scheme of the asset price process and the regression (25) are bounded. In particular, we have the following theorem.

Theorem 4.1: *Under the assumptions in Theorem 3.4, consider* $\bar{V}_{T_j} = v\left(T_j, X_{T_j}^{\bar{\pi},\bar{\mathcal{A}}}\right)$, $j = 1, 2, ..., \mathcal{J}$, *where* $X_{T_j}^{\bar{\pi},\bar{\mathcal{A}}}$ *is the Euler discretization of* X_{T_j} *corresponding to partitions* $\bar{\pi} \supset \pi$ *and* $\bar{\mathcal{A}} \supset \mathcal{A}$. *Let* $\bar{M}_{T_j}$ *be the martingale component of* $\bar{V}_{T_j}$. *Assuming that for* $0 \leq l \leq \mathcal{L} - 1$ *and* $1 \leq k \leq \mathcal{K}$,

$$\begin{cases} \|\hat{\phi}^{\pi}(t_l,\cdot) - \bar{\phi}^{\pi}(t_l,\cdot)\|_1^2 \leq \epsilon \\ \|\hat{\psi}^{\pi,\mathcal{A}}(t_l,y_k,\cdot) - \bar{\psi}^{\pi,\mathcal{A}}(t_l,y_k,\cdot)\|_1^2 \leq \epsilon \end{cases}$$

for some positive number ϵ, *then there exist positive constants* $\bar{C}, \bar{C}'$ *and* $\bar{C}''$ *such that*

$$E\left[\max_{0\leq j\leq \mathcal{J}} |\hat{M}_{T_j}^{\pi,\mathcal{A}} - \bar{M}_{T_j}|^2\right] \leq \bar{C}\cdot\frac{1}{\mathcal{L}} + \bar{C}'\cdot\frac{1}{K^2} + \bar{C}''\cdot\epsilon.$$

Proof: See Appendix B for proof. □

5. Numerical experiments

In this section, we will conduct numerical experiments to illustrate the computational efficiency of our proposed T-M algorithm on a Bermudan option pricing problem under a jump-diffusion model. The exact model we consider here falls into the class of jump-diffusion models (see Merton (1976) and Kou (2002)) reviewed in Section 1. Specifically, the asset prices evolve as follows:

$$\frac{dX(t)}{X(t^-)} = (r-\delta)\,dt + \sigma dW(t) + d\left(\sum_{i=1}^{P(t)}(V_i - 1)\right), \tag{29}$$

where r is the constant discount factor, δ is the constant dividend rate, σ is the constant volatility, $\{X(t) = [X_1(t), ..., X_n(t)]\}$ represents the asset price with a given initial price X_0, $\{W(t) = [W_1(t), ..., W_n(t)]\}$ is a Wiener process, $\{P(t)\}$ is a Poisson process with constant intensity λ, and $\{V_i\}$ is a sequence of independent identically distributed (i.i.d.) nonnegative random variables such that $J = \log(V)$ is the jump amplitude with p.d.f. $f(y)$. Here J can follow a normal distribution (see Merton (1976)) or a double-exponential distribution (see Kou (2008)) or various other reasonable distributions. For simplicity, we assume J follows a one-dimensional ($d = 1$) normal distribution $N(m, \theta^2)$. We also assume $W(t)$, $P(t)$ and J are mutually independent.

As explored in *(iii)* of Remark **1**, to connect dynamics (29) with the jump-diffusion model (1) we have mainly focused on, we should construct a function of $X(t)$, denoted by $S(t) = S(X(t))$, such that dynamics (29) can be easily transformed to an equivalent dynamics jointly driven by the Wiener measure and a Poisson random measure $\mathcal{P}_S$. The following proposition provides an intuitive criterion in selecting such a function by explicitly defining the intensity function $\mu(dt \times dy)$ for the unique Poisson random measure induced by a (non-homogenous) compound Poisson process.

Proposition 5.1: (Extension of Proposition 3.5 in Cont and Tankov (2003)) *Let* $S(t)_{t>0}$ *be a compound (non-homogenous) Poisson process with intensity* λ_t *and jump size distribution* f. *Then the Poisson random measure* $\mathcal{P}_S$ *induced by* $S(t)_{t>0}$ *on* $[0,\infty] \times \mathrm{R}^d$ *has intensity measure* $\mu(dt \times dy) = \lambda_t f(y)\,dydt$.

According to Proposition 5.1, for a compound Poisson process $S(t)$, the compensated Poisson random measure $\tilde{\mathcal{P}}_S$ induced by $S(t)$ can be simulated by $\tilde{\mathcal{P}}_S = \mathcal{P}_S - \lambda_t f(y)\, dydt$. Although $X(t)$ given by (29) is not a compound process, $S(t) = \log(X(t))$ is a compound Poisson process, and thus its Poisson random measure $\mathcal{P}_S(t, y)$ can be easily simulated according to Proposition 5.1. Specifically, we can incorporate $\mathcal{P}_S$ into the asset-price dynamics (29) and reformulate it as follows:

$$\frac{dX(t)}{X(t^-)} = (r - \delta)\, dt + \sigma dW(t) + \int_{\mathrm{R}^d} y \mathcal{P}_S(dt, dy). \tag{30}$$

Unfortunately, the solution to asset dynamics (29) or (30) is not uniquely determined in the risk-neutral sense, caused by the incompleteness of the market under the jump-diffusion setting. However, we can construct different pricing measures $\mathrm{Q}'s \sim \mathrm{P}$ such that the discounted price $\hat{X}(t)$ is a martingale under $\mathrm{Q}'s$ (cf. Chapter 10 in Cont and Tankov (2003)). Here we will adopt the construction method proposed by Merton (1976). That is, changing the drift of the Wiener process but leaving other components of (29) unchanged to offset the jump results in a risk-neutral measure Q_M, which is a generalization of the unique risk-neutral measure under the Black-Scholes model. Therefore, the solution under Q_M can be easily derived and efficiently simulated. Precisely, the solution to the asset-price dynamics (29) is given by:

$$X(t) = X_0 \exp\Big[\mu^M t + \sigma W^M(t) + \sum_{i=1}^{P(t)} J_i\Big], \quad t > 0, \tag{31}$$

where $\mu^M = r - \delta - \frac{1}{2}\sigma^2 - E\left[e^J - 1\right]$ is the new drift, $\{W^M(t)\}$ is a standard vector Wiener process and $J_i' s$ are the i.i.d. random variables according to J.

Given the equivalence of (29) and (31), we can simulate $X(t)$ under the risk-neutral measure Q_M by simulating $S(t)$ and its associated Poisson random measure $\mathcal{P}_S$. Specifically, we perform the Euler scheme on an equi-length partition $\bar{\pi}$ with $|\bar{\pi}| = 0.01$ and an equi-probability partition (w.r.t. $f(y)$) on R^d with connected subsets and $|\bar{\mathcal{A}}| = 0.1$ (which means $|\bar{\mathcal{K}}| = 10$) to simulate the Wiener increments $\{W_{t_l}\}$, the Poisson random measure increments $P([t_l, t_{l+1}] \times A_k)$, and the resulting sample paths of $X(t) = \exp(S(t))$ according to (30) and (31).

We consider a Bermudan Min-Puts on the n assets with price vector $\{X_1(t), ..., X_n(t)\}$. In particular, at any time $t \in \Xi = \{T_0, T_1, ..., T_{\mathcal{J}}\}$, the option holder has the right to exercise the option to receive the payoff

$$h(X(t)) = (SK - \min(X_1(t), ..., X_n(t)))^+,$$

where SK represents the strike price. The maturity time of the option is $T = 1$ and can be exercised at 11 equally-spaced time points, i.e., $T_j = j \times T/10,\ j = 0, ..., 10$. Our objective is to solve the Bermudan option pricing problem by providing a lower bound and an upper bound on the exact option price.

5.1 *Suboptimal exercise strategies and lower Bounds*

First, let us adopt the L-S algorithm to generate a suboptimal exercise strategy $\tilde{\tau}$ by regressing the continuation values, and compute the corresponding benchmark lower bound. It turns out this algorithm will be very effective if one can construct good function bases for regression in the sense that the function bases should capture sufficient features of the continuation values. In particular, under pure-diffusion models, Anderson and Broadie (2004) propose a function basis consisting of all monomials of underlying asset prices with degrees less than or equal to three, the European min-put option with maturity T, its square and its cube. Numerical tests show that this function basis works extremely well, because European option price under the pure-diffusion models can capture sufficient features of the Bermudan option, and be fast numerically evaluated via its closed-form.

For the Bermudan option pricing problem under the jump-diffusion model (29), the corresponding European option still has a closed-form expression. Specifically, the explicit form of the European option on X at time t with maturity T, denoted by $C^M(t, X; T)$, is given by:

$$C^M(t, X; T) = E_{\mathbb{Q}_M}\left[h(X_T)|\mathcal{F}_t\right] = \sum_{k\geq 0} \frac{e^{-\lambda(T-t)}\left(\lambda(T-t)\right)^k}{k!} C^{BS}_{\sigma_k}\left(0, X_k; T-t\right), \tag{32}$$

where $\sigma_k^2 = \sigma^2 + k\theta^2/\left(T-t\right)$, $X_k = X\exp\left(k(m+\frac{\theta^2}{2}) - \lambda[\exp(m+\frac{\theta^2}{2})-1](T-t)\right)$ and

$$\begin{aligned} C^{BS}_{\sigma}(0, X; \tau) &= E\left[H(Xe^{(r-\delta-\frac{\sigma^2}{2})\tau+\sigma W_\tau})\right] \\ &= -\sum_{l=1}^{n} X^l \frac{e^{-\delta\tau}}{\sqrt{2\pi}} \int_{(-\infty, d^l_-]} \exp\left(-\frac{z^2}{2}\right) \prod_{l'=1, l'\neq l}^{n} \mathcal{N}\left(\frac{\ln\frac{X^{l'}}{X^l}}{\sigma\sqrt{\tau}} - z - \sigma\sqrt{\tau}\right) dz \\ &\quad + e^{-r\tau}\cdot SK \cdot \left(1 - \prod_{l=1}^{n}\left(1 - \mathcal{N}\left(d^l_+\right)\right)\right), \end{aligned} \tag{33}$$

with

$$d^l_+ = \frac{\ln\frac{SK}{X^l} - \left(r - \delta - \frac{\sigma^2}{2}\right)\tau}{\sigma\sqrt{\tau}}, \qquad d^l_- = d^l_+ - \sigma\sqrt{\tau},$$

and $\mathcal{N}$ denoting the cumulative distribution function (c.d.f.) of a standard normal distribution, $C^{BS}_{\sigma}(0, X; \tau)$ denoting the European option under the Black-Scholes model with maturity τ, volatility σ and initial price X. Unfortunately, it is extremely difficult to compute $C^M(t, X; T)$ in (32) exactly because of the infinite sum. A natural modification is to approximate it numerically by a finite truncation of the sum in (32) and some approximation of the integrals in (33). However, for our case, we try to approximate it directly by an European option under a closely-related pure-diffusion model. Surprisingly, the most intuitive one with the simplest structure, i.e., the European option under the pure-diffusion model derived simply by removing the jump part of (29), works extremely well in our numerical experiments. To avoid confusion, we refer to it as "non-jump European option".

Now the function basis we choose includes all monomials of underlying asset prices with degrees less than or equal to three, the non-jump European option with maturity T, its square and its cube. With this basis, we implement the least-squares regression algorithm, and obtain suboptimal exercise strategies $\tilde{\tau}'s$ and the corresponding lower bounds, as shown in Table 1.

5.2 *Benchmark Upper Bounds*

After obtaining the suboptimal exercise strategies $\tilde{\tau}'s$, we adopt the A-B algorithm with nested Monte Carlo simulation to compute the benchmark upper bounds. We report the numerical results in Table 1, in which we can see that the A-B algorithm yields extremely tight upper bounds with small duality gaps. This observation indicates three facts. First, the suboptimal exercise strategies $\tilde{\tau}'s$ constructed with the function basis proposed in solving the primal problem are nearly optimal, which is crucial for the successful implementation of the T-M algorithm because the regression coefficients $\hat{\alpha}$ and $\hat{\beta}$ are obtained with employing $\tilde{\tau}'s$. Second, this function basis is used as the starting point for constructing the function bases ρ^W and $\rho^{\mathcal{P}}$ in solving the dual problem, therefore the effectiveness of this function basis is a strong indicator for the sufficiency of bases ρ^W and $\rho^{\mathcal{P}}$ in terms of capturing features. Finally, the A-B algorithm is still very effective under the jump-diffusion models, regardless of the considerable computational effort of the nested simulation.

5.3 *Upper Bound by True Martingale Approach*

Next, we will address the computational inefficiency suffered by the A-B algorithm by implementing our proposed T-M algorithm (Algorithm **2**) described in Section 4. Notice that we have addressed almost all the implementation details except the choices of the partitions π and $\mathcal{A}$, and the bases ρ^W and $\rho^{\mathcal{P}}$.

First of all, the choice of the partition π is essential to balance the tradeoff between the quality of the true martingale approximation and the computational efficiency (see Theorem 3.4). It has to be sufficiently fine to reduce the overall mean square error between the true martingale approximation and the objective martingale, but not too fine so that the computational effort for obtaining martingale approximation $M^{\pi,\mathcal{A}}$ is much less than the computational effort for exercising the strategy along all the inner sample paths in the A-B algorithm. In fact, a good way to achieve this tradeoff is to perform a regression-interpolation technique on a two-layer coarse-fine partition, i.e., performing the regression on the coarse partition and interpolating the regression coefficients piecewise constant across all the partitioning points of the finer partition. To maximize the effect of this regression-interpolation technique, we choose to perform the regression procedure on the original exercisable dates $\Xi = \{T_0, T_1, ..., T_{\mathcal{J}} - 1\}$ and interpolate the regression coefficients piecewise constant across all the partitioning points of the Euler scheme partition $\bar{\pi}$.

Secondly, as mentioned in (iv) of Remark **2**, a very coarse partition $\mathcal{A}$ is sufficient in capturing most of the jump size realizations. For the sake of convenience, we let $\mathcal{A} = \bar{\mathcal{A}}$ (which means $|\bar{\mathcal{K}}| = |\mathcal{K}| = 10$, amplifying the computational complexity by a factor of 10). Therefore, the compensated Poisson increments $\{\tilde{P}([t_l, t_{l+1}] \times A_k)\}$ in (25) are obtained immediately from the simulation of $X^{\bar{\pi},\bar{\mathcal{A}}}$, and $\mu([t_l, t_{l+1}] \times A_k)$ in (25) equals $\lambda \times 0.01 \times 0.1$ (see Proposition 5.1). Specifically, we obtain $\{\hat{\alpha}_{T_j}, j = 0, ..., \mathcal{J} - 1\}$ and $\{\hat{\beta}_{T_j,k}, j = 0, ..., \mathcal{J} - 1, k = 1, ..., \mathcal{K}\}$ via the regression (25), and set $\hat{\alpha}_{t_l} = \hat{\alpha}_{T_j}$ for $t_l \in [T_j, T_{j+1})$ and $\hat{\beta}_{t_l,k} = \hat{\beta}_{T_j,k}$ for $t_l \in [T_j, T_{j+1}), k = 1, ..., \mathcal{K}$.

Finally, the choice of the bases ρ^W and $\rho^{\mathcal{P}}$ affects the accuracy of the martingale approximation $\hat{M}^{\pi,\mathcal{A}}$. According to Theorem 3.2 and the following Remark **1**, the bases ρ^W should capture sufficient features of the derivative of the Bermudan option price, while the bases $\rho^{\mathcal{P}}$ should capture sufficient features of the increment of the Bermudan option price caused by the jump. Therefore, the most natural candidates to be considered in ρ^W and $\rho^{\mathcal{P}}$ are the derivative of the corresponding European option and the increment of the European option caused by the jump, respectively. To see the exact structure of the candidates, let us apply Theorem 3.2 w.r.t. the Wiener measure W^M and Poisson random measure $\mathcal{P}_S$ on the martingale part, denoted by $M^M(t, X_t, T_j)$, of the European option $C^M(t, X_t, T_j)$ for $0 \le j \le \mathcal{J}$, i.e.,

$$M^M(t, X_t; T_j) = \int_t^{T_j} \frac{\partial C^M(u, X_{u^-}; T_j)}{\partial X} X_{u^-} \sigma dW_u^M - \int_t^{T_j} \int_{\mathbb{R}^d} \left[C^M(u, X_{u^-} \cdot e^y; T_j) - C^M(u, X_{u^-}; T_j)\right] \tilde{\mathcal{P}}_S(du, dy).$$

Our preliminary numerical tests show that, for $t \in [T_j, T_{j+1})$ and $1 \le k \le \mathcal{K}$,

$$\rho^W(t, X_{t^-}) = \{1, \frac{\partial C^M(t, X_{t^-}; T_{j+1})}{\partial X} X_{t^-}, \frac{\partial C^M(t, X_{t^-}; T)}{\partial X} X_{t^-}\},$$

and

$$\rho^{\mathcal{P}}(t, y_k, X_{t^-}) = \{1, C^M(t, X_{t^-} \cdot e^{y_k}; T_{j+1}) - C^M(t, X_{t^-}; T_{j+1}), C^M(t, X_{t^-} \cdot e^{y_k}; T) - C^M(t, X_{t^-}; T)\}$$

Table 1. Bounds (with 95% confidence intervals) for Bermudan Min-put options

n	λ	X_0	Lower Bound (L-S algorithm)	Upper Bound (T-M algorithm)	Benchmark U-B (A-B algorithm)	CPU Time Ratio (T-M vs A-B)
1	1	36	5.842 ± 0.031	5.970 ± 0.031	5.899 ± 0.038	$\approx$ 1:400
1	1	40	3.791 ± 0.028	3.910 ± 0.033	3.856 ± 0.036	$\approx$ 1:400
1	1	44	2.383 ± 0.024	2.443 ± 0.028	2.417 ± 0.033	$\approx$ 1:400
1	3	36	7.702 ± 0.043	7.899 ± 0.030	7.810 ± 0.053	$\approx$ 1:400
1	3	40	5.817 ± 0.039	5.996 ± 0.047	5.894 ± 0.050	$\approx$ 1:400
1	3	44	4.352 ± 0.036	4.480 ± 0.044	4.440 ± 0.040	$\approx$ 1:400
2	1	36	8.133 ± 0.033	8.308 ± 0.045	8.243 ± 0.040	$\approx$ 1:9
2	1	40	5.691 ± 0.034	5.785 ± 0.040	5.755 ± 0.043	$\approx$ 1:9
2	1	44	3.765 ± 0.028	3.842 ± 0.036	3.804 ± 0.038	$\approx$ 1:9
2	3	36	9.786 ± 0.045	10.038 ± 0.061	9.989 ± 0.057	$\approx$ 1:9
2	3	40	7.680 ± 0.043	7.900 ± 0.060	7.845 ± 0.057	$\approx$ 1:9
2	3	44	5.941 ± 0.040	6.118 ± 0.058	6.065 ± 0.058	$\approx$ 1:9

The payoff of the min-put option is: $(SK - min(X_1(t), ..., X_n(t)))^+$. The parameters are: $SK = 40, r = 4\%, \delta = 0, \sigma = 20\%, m = 6\%, \theta = 20\%, T = 1, \mathcal{J} = 10$. The jump intensity λ is 1 or 3 and the initial price is $X_0 = (X, ..., X)$ with X =36, 40 or 44, as shown in the table. We use $N = 5 \times 10^4$ sample paths to estimate the regression coefficients to determine the suboptimal exercise strategy, and we use $N = 5 \times 10^4$ sample paths to estimate the coefficients $\hat{\alpha}$ and $\hat{\beta}$. We use $N_1 = 10^5$ sample paths to determine the lower bounds. For the implementation of the A-B algorithm, we use $N_2 = 10^3$ outer sample paths and $N_3 = 5 \times 10^2$ inner sample paths to determine the benchmark upper bounds and the confidence intervals of appropriate length. For the implementation of the T-M algorithm, we use $\bar{N} = 2.5 \times 10^3$ sample paths to determine the true upper bounds and the confidence intervals of appropriate length.

yield desirable upper bounds, where $C^M(t, \cdot, \cdot)$ is approximated via a finite truncation of the infinite sum in (32) and $y_k \in A_k$ is a representative value. More importantly, we observe that the 1^{st} order truncation and the 2^{nd} order truncation yield upper bounds with random negligible differences. Note that the truncation approximations are merely weighted sums of European option prices under certain Black-Scholes models. In fact, in our formal numerical tests, we approximate $C^M(t, \cdot, \cdot)$ directly by the corresponding "non-jump European option" $C^{BS}(t, \cdot, \cdot)$ (see Bases 4 in Table 3), in view of its simple structure. It turns out the upper bounds generated by this particular choice of bases are as good as the upper bounds generated by the bases approximated by finite truncation.

We report the numerical results on the lower bounds by the L-S algorithm, the benchmark upper bounds by the A-B algorithm and the true upper bounds by the T-M algorithm in Table 1. Below are several key observations. First, the small gaps between the lower bounds and the true upper bounds indicate that the T-M algorithm is quite effective in terms of generating tight true upper bounds. Second, the small confidence intervals of the true upper bounds indicate that T-M algorithm generates good approximations of the optimal dual martingales. Finally, note that the A-B algorithm generates slightly tighter upper bounds than our proposed T-M algorithm; however, the CPU time ratios indicate that T-M algorithm achieves a much higher numerical efficiency.

It is instructive to theoretically compare the computational complexity of the T-M algorithm and the A-B algorithm, since the CPU time ratios in Table 1 are quite different for 1-dimensional problems and 2-dimensional problems.

We use $t_{EE} + t_{ES}$ to represent the total CPU time, where t_{EE} represents the CPU time for evaluating the European option prices and t_{ES} represents the CPU time for exercising the strategy along all the sample paths. When $n = 1$, simple numerical tests show that t_{EE} is negligible and t_{ES} dominates the total CPU time. Indeed, when $n = 1$, evaluating the basis functions–European option prices–reduces to evaluating the c.d.f. of a standard normal distribution which is immediately available in most computer software (e.g. Matlab). Therefore, the CPU time ratio

Table 2. Upper Bounds for different function bases

λ	X_0	Bases 1	Bases 2	Bases 3	Bases 4
1	36	6.730 ± 0.069	6.283 ± 0.042	6.228 ± 0.048	5.970 ± 0.031
1	40	4.789 ± 0.074	4.228 ± 0.039	4.127 ± 0.047	3.910 ± 0.033
1	44	3.344 ± 0.073	2.734 ± 0.038	2.665 ± 0.044	2.443 ± 0.028
3	36	8.829 ± 0.091	8.338 ± 0.059	8.167 ± 0.062	7.899 ± 0.030
3	40	7.086 ± 0.101	6.377 ± 0.060	6.277 ± 0.067	5.996 ± 0.047
3	44	5.681 ± 0.100	4.953 ± 0.057	4.752 ± 0.061	4.480 ± 0.044

Table 3. Explicit description of function Bases in Table 5.2

Bases	$\rho^W(t,x)$ with $t \in [T_{j-1}, T_j)$	$\rho^{\mathcal{P}}(t, y_k, x)$ with $t \in [T_{j-1}, T_j)$, $1 \le k \le \mathcal{K}$
Bases 1	$\{1\}$	$\{1\}$
Bases 2	$\{1,\ x,\ x^2,\ x^3\}$	$\{1,\ x,\ x^2,\ x^3\}$
Bases 3	$\{1,\ C^{BS}(t,x;T),\ \left(C^{BS}(t,x;T)\right)^2\}$	$\{1,\ C^{BS}(t,x;T),\ \left(C^{BS}(t,x;T)\right)^2\}$
Bases 4	$\{1,\ \frac{\partial C^{BS}(t,x;T_{j+1})}{\partial x}x,\ \frac{\partial C^{BS}(t,x;T)}{\partial x}x\}$	$\{1, C^{BS}(t, x\cdot e^{y_k}; T_{j+1}) - C^{BS}(t,x;T_{j+1}), C^{BS}(t, x\cdot e^{y_k}; T) - C^{BS}(t,x;T)\}$

is in the order of the ratio of # sample paths, which is consistent with the numerical result ($\approx$ 1:400). However, when $n \ge 2$, t_{EE} is no long negligible and starts to dominate the total CPU time, because evaluating the European option prices now becomes evaluating the integrals of the c.d.f. of a standard normal distribution, which consumes a significant amount of time. In fact, simple numerical tests show that t_{EE} consumes 95% of the total CPU time. Therefore, to estimate the CPU time ratio, we should compare the total evaluation times of the European option prices for both algorithms. For the A-B algorithm, the total evaluation times is in the order of $(N_2 \times N_3 \times \mathcal{J}^2)$, which is quadratic in the number of exercisable periods $\mathcal{J}$ and will be significantly amplified by the nested simulation. For the T-M algorithm, the total evaluation times is in the order of $(\bar{N} \times \mathcal{L} \times \mathcal{K})$, which is linear in the number of exercisable periods $\mathcal{J}$, since $\mathcal{L}$ is a linear function of $\mathcal{J}$ (in our case $\mathcal{L} = 10\mathcal{J}$) and $\mathcal{K}$ is usually quite small (in our case $\mathcal{K} = 10$). Therefore, we can estimate the order of CPU time ratio by $(\bar{N} \times \mathcal{L} \times \mathcal{K}) : (N_2 \times N_3 \times \mathcal{J}^2)$, which is consistent with the numerical result ($\approx$ 1:9). Overall, T-M algorithm can achieve a higher order of computational efficiency. Furthermore, we can expect the CPU time ratios (T-M algorithm versus A-B algorithm) to further decrease when the number of exercisable periods increases, and remain stable if the dimension of the problem increases.

An interesting experiment has been conducted to exhibit the quality differences between the upper bounds generated by different bases ρ^W and $\rho^{\mathcal{P}}$, and we report the results in Table 2. Specifically, the upper bounds of different levels of quality are generated using various bases that are presented in Table 3. To summarize, the simplest basis, i.e., $\{1\}$ (Bases 1), results in very poor upper bounds; the standard basis one can come up with, i.e., the polynomials (Bases 2), improves the upper bounds significantly, but the gap is still too large. However, the upper bounds get almost no improvement after we use the non-jump European options as the basis (Bases 3), which indicates that the non-jump European option does not further provide useful features. Finally, the basis ρ^W consisting of the deltas of the non-jump European options and the basis $\rho^{\mathcal{P}}$ consisting of the non-jump European option increments (Bases 4) yield desirable upper bounds. These results verify the theoretical analysis about the structure of the optimal dual martingales, as shown in Theorem 3.2.

Another interesting experiment has been conducted to investigate the individual performance of the two terms in (27), since each term individually is a well-defined true martingale (adapted

Table 4. Upper Bounds by one term in the True Martingale

λ	X_0	Term 1	Term 2	Complete Martingale
1	36	6.863 ± 0.059	7.930 ± 0.073	5.970 ± 0.031
1	40	4.450 ± 0.056	5.184 ± 0.072	3.910 ± 0.033
1	44	2.750 ± 0.050	3.125 ± 0.064	2.443 ± 0.028
3	36	10.101 ± 0.099	9.304 ± 0.070	7.899 ± 0.030
3	40	7.776 ± 0.103	7.047 ± 0.070	5.996 ± 0.047
3	44	5.747 ± 0.098	5.244 ± 0.066	4.480 ± 0.044

to the filtration $\{\mathcal{F}_{T_j} : j = 0, ..., \mathcal{J}\}$). Specifically, if

$$\begin{aligned}\hat{M}^{\pi,\mathcal{A}}_{T_j} &= \sum_{0 \le t_l < T_j} \hat{\phi}^{\pi}\left(t_l, X^{\bar{\pi},\bar{\mathcal{A}}}_{t_l}\right)(\Delta^{\pi} W_l) + \sum_{0 \le t_l < T_j} \sum_{k=1}^{\mathcal{K}} \hat{\psi}^{\pi,\mathcal{A}}\left(t_l, y_k, X^{\bar{\pi},\bar{\mathcal{A}}}_{t_l}\right) \tilde{P}\left([t_l, t_{l+1}] \times A_k\right) \\ &= (Term\ 1)\ +\ (Term\ 2),\end{aligned}$$

then both Term 1 and Term 2 are martingales adapted to the filtration $\{\mathcal{F}_{T_j} : j = 0, ..., \mathcal{J}\}$. Therefore both of them will induce true upper bounds. Results in Table 4 show that taking out either one of these two terms yields significantly poorer upper bounds with much worse confidence intervals, which implies that the effort we have spent on the regression coefficients $\hat{\alpha}$ and $\hat{\beta}$, and the construction of the martingale $M^{\pi,\mathcal{A}}$ is necessary and time-worthy.

6. Conclusion and Future directions

In this paper, we extend the true martingale approach proposed by Belomestny *et al.* (2009) for the pure-diffusion models to the jump-diffusion models, to fast compute true tight upper bounds on Bermudan option price. It is a useful alternative of the classic A-B algorithm proposed by Anderson and Broadie (2004), especially when the computational budget is limited. The theoretical analysis proves and numerical results verify that our algorithm generates stable and tight upper bounds with significant reduction of the computational effort. Moreover, we explore the structure of the optimal dual martingale for the dual problem and provide an intuitive understanding towards the construction of good approximations of the optimal dual martingale over the space of all adapted martingales.

Furthermore, from the information relaxation point of view (see Brown *et al.* (2010)), we can gain an intuitive understanding towards the structure of the optimal penalty function. It inspires us to construct good penalty functions over the space of "feasible penalty functions" for general dynamic programming problems, which is still an open area to explore (see Ye and Zhou (2012) for some initial exploration).

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Appendix A: Proof of Theorem 3.4

Proof: Fix T_j. Consider t_l such that $T_j \leq t_l < T_{j+1}$ and k such that $1 \leq k \leq \mathcal{K}$. According to (16) and (17), we have:

$$\begin{aligned}
\phi^{\pi}_{t_l} &= \frac{1}{\Delta^{\pi}_l} E_{t_l}\left[\left(\Delta^{\pi} W_l\right) V_{T_{j+1}}\right] \\
&\overset{(i)}{=} \frac{1}{\Delta^{\pi}_l} E_{t_l}\left[\left(\Delta^{\pi} W_l\right)\left(V_{T_{j+1}} - E_{T_j}[V_{T_{j+1}}]\right)\right] \\
&\overset{(ii)}{=} \frac{1}{\Delta^{\pi}_l} E_{t_l}\left[\left(\Delta^{\pi} W_l\right)\left(M_{T_{j+1}} - M_{T_j}\right)\right] \\
&\overset{(iii)}{=} \frac{1}{\Delta^{\pi}_l} E_{t_l}\left[\left(\int_{t_l}^{t_{l+1}} dW_s\right)\left(\int_{T_j}^{T_{j+1}} \phi_s dW_s + \int_{T_j}^{T_{j+1}}\int_{\mathrm{R}^d} \psi\left(s,y\right)\tilde{\mathcal{P}}\left(ds,dy\right)\right)\right] \\
&\overset{(iv)}{=} \frac{1}{\Delta^{\pi}_l} E_{t_l}\left[\int_{t_l}^{t_{l+1}} \phi_s ds\right],
\end{aligned}$$

where the equality (i) follows from the independence between the Wiener increment $\Delta^{\pi} W_l$ and the conditional value function $E_{T_j}[V_{T_{j+1}}]$, as well as the zero-mean property of $\Delta^{\pi} W_l$, equality (ii) uses (8), equality (iii) uses (11), and equality (iv) follows from the Ito's isometry and the independence between W and $\mathcal{P}$.

Similarly, we have

$$\begin{aligned}
\psi^{\pi,\mathcal{A}}\left(t_l, y_k\right) &= \frac{1}{\mu\left(\left[t_l, t_{l+1}\right] \times A_k\right)} E_{t_l}\left[\tilde{\mathcal{P}}\left(\left[t_l, t_{l+1}\right] \times A_k\right) V_{T_{j+1}}\right] \\
&\stackrel{(i)}{=} \frac{1}{\mu\left(\left[t_l, t_{l+1}\right] \times A_k\right)} E_{t_l}\left[\tilde{\mathcal{P}}\left(\left[t_l, t_{l+1}\right] \times A_k\right)\left(V_{T_{j+1}} - E_{T_j}\left[V_{T_{j+1}}\right]\right)\right] \\
&\stackrel{(ii)}{=} \frac{1}{\mu\left(\left[t_l, t_{l+1}\right] \times A_k\right)} E_{t_l}\left[\tilde{\mathcal{P}}\left(\left[t_l, t_{l+1}\right] \times A_k\right)\left(M_{T_{j+1}} - M_{T_j}\right)\right] \\
&\stackrel{(iii)}{=} \frac{1}{\mu\left(\left[t_l, t_{l+1}\right] \times A_k\right)} E_{t_l}\left[\left(\int_{t_l}^{t_{l+1}}\int_{A_k}\tilde{\mathcal{P}}\left(ds, dy\right)\right)\left(\int_{T_j}^{T_{j+1}}\phi_s dW_s + \int_{T_j}^{T_{j+1}}\int_{\mathrm{R}^d}\psi\left(s, y\right)\tilde{\mathcal{P}}\left(ds, dy\right)\right)\right] \\
&\stackrel{(iv)}{=} \frac{1}{\mu\left(\left[t_l, t_{l+1}\right] \times A_k\right)} E_{t_l}\left[\int_{t_l}^{t_{l+1}}\int_{A_k}\psi\left(s, y\right)\mu\left(ds \times dy\right)\right],
\end{aligned}$$

where equality (i) follows from the independence between the compensated Poisson increment $\tilde{\mathcal{P}}\left(\left[t_l, t_{l+1}\right] \times A_k\right)$ and the conditional value function $E_{T_j}[V_{T_{j+1}}]$, as well as the zero-mean property of $\tilde{\mathcal{P}}\left(\left[t_l, t_{l+1}\right] \times A_k\right)$, equality (ii) uses (8), equality (iii) uses (11), and equality (iv) follows from Ito's isometry and the independence between $\mathcal{P}$ and W.

Furthermore, from (8) and (11), we have:

$$\begin{aligned}
V_{T_{j+1}} - E_{T_j}\left[V_{T_{j+1}}\right] &= M_{T_{j+1}} - M_{T_j} \\
&= \int_{T_j}^{T_{j+1}}\phi_s dW\left(s\right) + \int_{T_j}^{T_{j+1}}\int_{\mathrm{R}^d}\psi\left(s, y\right)\tilde{\mathcal{P}}\left(ds, dy\right).
\end{aligned} \tag{A1}$$

If we reorganize (1) and (A1) together as the following Forward-Backward SDE (FBSDE) on $[T_j, T_{j+1}]$

$$\begin{cases} X_t = X_{T_j} + \int_{T_j}^t b(s, X_s)ds + \int_{T_j}^t \sigma(s, X_s)dW_s + \int_{T_j}^t\int_{\mathrm{R}^d} J\left(X\left(s\right), y\right)\tilde{\mathcal{P}}\left(ds, dy\right) \\ Y_t := v\left(T_{j+1}, X_{T_{j+1}}\right) - \int_t^{T_{j+1}}\phi_s dW_s - \int_t^{T_{j+1}}\int_{\mathrm{R}^d}\psi\left(s, y\right)\tilde{\mathcal{P}}\left(ds, dy\right) \end{cases},$$

then according to Theorem 2.1 in Bouchard and Elie (2008), we have

$$E\left[\sum_{T_j \le t_l < T_{j+1}}\int_{t_l}^{t_{l+1}}|\phi_s - \phi_{t_l}^{\pi,\mathcal{A}}|^2 ds\right] \le C_j|\pi|, \tag{A2}$$

for some constant C_j.

According to Corollary 4.1 in Bouchard and Elie (2008), we have

$$E\left[\sup_{r\in[s,t]}|\psi(r, y) - \psi(s, y)|^2\right] \le \tilde{C}|t - s|, \quad \forall y \in \mathrm{R}^d, \tag{A3}$$

for some constant $\tilde{C}$. Furthermore, According to Remark 4.1 and Proposition 4.4 in Bouchard and Elie (2008), $\forall s \in [t_l, t_{l+1}], y \in A_k, \tilde{y} \in A_k$, we have

$$|\psi(s, y) - \psi(s, \tilde{y})|^2 \le \tilde{C}|J\left(X_{s^-}, y\right) - J\left(X_{s^-}, \tilde{y}\right)|^2 \le \tilde{C}'\|y - \tilde{y}\|^2,$$

where we recall $J(\cdot, \cdot)$ is defined in (1) in Section 2.2 and assumed to be Lipschitz continuous. Therefore, $\forall s \in [t_l, t_{l+1}], \tilde{s} \in [t_l, t_{l+1}], y \in A_k, \tilde{y} \in A_k$,

$$E\left[|\psi(s, y) - \psi(\tilde{s}, \tilde{y})|^2\right] \le \tilde{C}|t_{l+1} - t_l| + \tilde{C}'\|y - \tilde{y}\|^2. \tag{A4}$$

Hence, $\forall s \in [t_l, t_{l+1}], y \in A_k$,

$$
\begin{aligned}
& E\left[|\psi(s,y) - \psi^{\pi,\mathcal{A}}(t_l, y_k)|^2\right] \\
= & \ E\left[E_{t_l}\left[|\psi(s,y) - \frac{1}{\mu([t_l, t_{l+1}] \times A_k)} \int_{t_l}^{t_{l+1}} \int_{A_k} \psi(\tilde{s}, \tilde{y}) \mu(d\tilde{s} \times d\tilde{y})|^2\right]\right] \\
= & \ E\left[E_{t_l}\left[|\frac{1}{\mu([t_l, t_{l+1}] \times A_k)} \int_{t_l}^{t_{l+1}} \int_{A_k} (\psi(s,y) - \psi(\tilde{s}, \tilde{y})) \mu(d\tilde{s} \times d\tilde{y})|^2\right]\right] \\
\overset{(i)}{\leq} & \ \frac{1}{\mu([t_l, t_{l+1}] \times A_k)} E\left[E_{t_l}\left[\int_{t_l}^{t_{l+1}} \int_{A_k} |\psi(s,y) - \psi(\tilde{s}, \tilde{y})|^2 \mu(d\tilde{s} \times d\tilde{y})\right]\right] \\
\overset{(ii)}{\leq} & \ \tilde{C}|t_{l+1} - t_l| + \tilde{C}' \frac{1}{|A_k|} \int_{A_k} \|y - \tilde{y}\|^2 \lambda(d\tilde{y}),
\end{aligned}
\tag{A5}
$$

where inequality (i) follows from Jensen's Inequality and inequality (ii) follows from (A3) and (A4). To control the error term in (A5), let us choose an equi-length partition π on the time space. For the partition $\mathcal{A}$ on the jump space R^d, define the last partitioning set $A_{\mathcal{K}}$ as$A_{\mathcal{K}} = \{y \in \mathrm{R}^d : \|y\| > \frac{K}{2}\}$, then perform a $\overbrace{K^2 \times K^2 \times \cdots \times K^2}^{d}$ equi-distance grid on $\mathrm{R}^d \setminus A_{\mathcal{K}}$ to obtain K^{2d} partitioning sets $\{A_k : 1 \leq k \leq K^{2d}\}$ with $\|y - \tilde{y}\| \leq \frac{1}{K}, \forall y, \tilde{y} \in A_k \ \forall k = 1, ..., K^{2d}$. Now we have, $\forall s \in [t_l, t_{l+1}], y \in A_k$, $1 \leq k \leq K^{2d}$,

$$
\begin{aligned}
E\left[|\psi(s,y) - \psi^{\pi,\mathcal{A}}(t_l, y_k)|^2\right] &\leq \tilde{C}|t_{l+1} - t_l| + \tilde{C}' \frac{1}{|A_k|} \int_{A_k} \|y - \tilde{y}\|^2 \lambda(d\tilde{y}) \\
&\leq \tilde{C} \cdot \frac{1}{\mathcal{L}} + \tilde{C}' \cdot \frac{1}{K^2}.
\end{aligned}
\tag{A6}
$$

Similarly, $\forall s \in [t_l, t_{l+1}], y \in A_{\mathcal{K}}$,

$$
E\left[|\psi(s,y) - \psi^{\pi,\mathcal{A}}(t_l, y_{\mathcal{K}})|^2\right] \leq \tilde{C} \cdot \frac{1}{\mathcal{L}} + \tilde{C}' \frac{1}{|A_{\mathcal{K}}|} \int_{A_{\mathcal{K}}} \|y - \tilde{y}\|^2 \lambda(d\tilde{y}). \tag{A7}
$$

Therefore,

$$
\begin{aligned}
& E\left[\sum_{T_j \leq t_l < T_{j+1}} \sum_{k=1}^{\mathcal{K}} \int_{t_l}^{t_{l+1}} \int_{A_k} |\psi(s,y) - \psi^{\pi,\mathcal{A}}(t_l, y_k)|^2 \mu(ds \times dy)\right] \\
& \leq \mu\left([T_j, T_{j+1}] \times \mathrm{R}^d\right) \tilde{C} \cdot \frac{1}{\mathcal{L}} + \mu\left([T_j, T_{j+1}] \times \mathrm{R}^d\right) \tilde{C}' \cdot \frac{1}{K^2} \\
& \quad + |T_{j+1} - T_j| \tilde{C}' \frac{1}{|A_{\mathcal{K}}|} \int_{A_{\mathcal{K}}} \int_{A_{\mathcal{K}}} \|y - \tilde{y}\|^2 \lambda(d\tilde{y}) \lambda(dy) \\
& \triangleq \tilde{C}_j \cdot \frac{1}{\mathcal{L}} + \tilde{C}_j' \cdot \frac{1}{K^2} + \tilde{\delta}_j(K),
\end{aligned}
\tag{A8}
$$

where note that $\tilde{\delta}_j(K) \to 0$ as $K \to \infty$. Therefore, we have

$$
\begin{aligned}
&E\left[\max_{0\le j\le \mathcal{J}} |M^{\pi,\mathcal{A}}_{T_j} - M_{T_j}|^2\right] \overset{(i)}{\le} 4E\left[|M^{\pi,\mathcal{A}}_{T} - M_{T}|^2\right] \\
&= 4E\Big[\Big|\sum_{j=0}^{\mathcal{J}-1}\Big(\sum_{T_j\le t_l<T_{j+1}} \int_{t_l}^{t_{l+1}} \left(\phi_t - \phi^{\pi,\mathcal{A}}_{t_l}\right) dW\left(t\right) \\
&\quad + \sum_{T_j\le t_l<T_{j+1}} \sum_{k=1}^{\mathcal{K}} \int_{t_l}^{t_{l+1}} \int_{A_k} \left(\psi\left(t,y\right) - \psi^{\pi,\mathcal{A}}\left(t_l, y_k\right)\right) \tilde{\mathcal{P}}\left(dt,dy\right)\Big)\Big|^2\Big] \\
&\overset{(ii)}{=} 4\sum_{j=0}^{\mathcal{J}-1}\Big(E\left[\sum_{T_j\le t_l<T_{j+1}} \int_{t_l}^{t_{l+1}} |\phi_s - \phi^{\pi,\mathcal{A}}_{t_l}|^2 ds\right] \\
&\quad + E\left[\sum_{T_j\le t_l<T_{j+1}} \sum_{k=1}^{\mathcal{K}} \int_{t_l}^{t_{l+1}} \int_{A_k} |\psi(s,y) - \psi^{\pi,\mathcal{A}}\left(t_l, y_k\right)|^2 \mu\left(ds\times dy\right)\right]\Big) \\
&\le C\cdot\frac{1}{\mathcal{L}} + C'\cdot\frac{1}{K^2} + \delta(K),
\end{aligned}
\tag{A9}
$$

where the inequality (i) follows from Doob's L^p maximal inequality, equality (ii) follows from Ito's isometry and the independence between $\mathcal{P}$ and W, and $C = 4\sum_{j=0}^{\mathcal{J}-1}(C_j + \tilde{C}_j)$, $C' = 4\sum_{j=0}^{\mathcal{J}-1}\tilde{C}'_j$, $\delta(K) = 4\sum_{j=0}^{\mathcal{J}-1}\tilde{\delta}_j(K)$, and note that $\delta(K) \to 0$ as $K \to \infty$.

Now further assume the tail of λ's second moment, i.e. $\int_{\|y\|>\frac{K}{2}} \|y\|^2\lambda(dy)$, decays "at least" quadratically in K. From (A8), we know that

$$
\begin{aligned}
\delta(K) &= 4\sum_{j=0}^{\mathcal{J}-1}\tilde{\delta}_j(K) \\
&= 4T\tilde{C}'\frac{1}{|A_{\mathcal{K}}|}\int_{A_{\mathcal{K}}}\int_{A_{\mathcal{K}}} \|y-\tilde{y}\|^2\lambda(d\tilde{y})\lambda(dy) \\
&= 4T\tilde{C}'\left(\int_{\|y\|>\frac{K}{2}} \|y\|^2\lambda(dy) - \frac{\left(\int_{\|y\|>\frac{K}{2}}\|y\|\lambda(dy)\right)^2}{\int_{\|y\|>\frac{K}{2}}\lambda(dy)}\right) \\
&\le 4T\tilde{C}'\int_{\|y\|>\frac{K}{2}} \|y\|^2\lambda(dy).
\end{aligned}
\tag{A10}
$$

Hence, $\delta(K)$ decays "at least" quadratically in K, and (A9) can be further strengthened to

$$
E\left[\max_{0\le j\le \mathcal{J}} |M^{\pi,\mathcal{A}}_{T_j} - M_{T_j}|^2\right] \le C\cdot\frac{1}{\mathcal{L}} + \hat{C}\cdot\frac{1}{K^2}
$$

for some constant $\hat{C}$. □

Appendix B: Proof of Theorem 4.1

Proof: To avoid confusion of notations, we denote

$$\bar{M}^{\pi,\mathcal{A}}_{T_j} := \sum_{0\le t_l<T_j} \bar{\phi}^{\pi}_{t_l}\left(\Delta^{\pi} W_l\right) + \sum_{0\le t_l<T_j}\sum_{k=1}^{\mathcal{K}} \bar{\psi}^{\pi,\mathcal{A}}\left(t_l, y_k\right) \tilde{P}\left([t_l, t_{l+1}] \times A_k\right).$$

Then, we have

$$\begin{aligned}
&E\left[\max_{0\le j\le \mathcal{J}} |\hat{M}^{\pi,\mathcal{A}}_{T_j} - \bar{M}_{T_j}|^2\right] \overset{(i)}{\le} 4E\left[|\hat{M}^{\pi,\mathcal{A}}_{T} - \bar{M}_{T}|^2\right]\\
\overset{(ii)}{\le}& 16E\left[|\hat{M}^{\pi,\mathcal{A}}_{T} - \bar{M}^{\pi,\mathcal{A}}_{T}|^2 + |\bar{M}^{\pi,\mathcal{A}}_{T} - M^{\pi,\mathcal{A}}_{T}|^2 + |M^{\pi,\mathcal{A}}_{T} - M_{T}|^2 + |M_T - \bar{M}_T|^2\right]\\
=&16[(*) + (**) + (***) + (****)],
\end{aligned}$$

where inequality (i) follows from Doob's L^p maximal inequality and inequality (ii) follows from Cauchy's inequality. From the assumption, we have

$$\begin{aligned}
(*) &= E\left[|\hat{M}^{\pi,\mathcal{A}}_{T} - \bar{M}^{\pi,\mathcal{A}}_{T}|^2\right]\\
&= \sum_{j=0}^{\mathcal{J}-1}\Bigg(E\left[\sum_{T_j\le t_l<T_{j+1}} \int_{t_l}^{t_{l+1}} \left(|\hat{\phi}^{\pi}_{t_l} - \bar{\phi}^{\pi}_{t_l}|^2\right) ds\right]\\
&\quad + E\left[\sum_{T_j\le t_l<T_{j+1}} \sum_{k=1}^{\mathcal{K}} \int_{t_l}^{t_{l+1}} \int_{A_k} \left(|\hat{\psi}^{\pi,\mathcal{A}}\left(t_l, y_k\right) - \bar{\psi}^{\pi,\mathcal{A}}\left(t_l, y_k\right)|^2\right) \mu\left(ds \times dy\right)\right]\Bigg)\\
&\le \left(\mu\left([0,T] \times \mathrm{R}^d\right) + T\right)\epsilon.
\end{aligned}$$

From Theorem 3.4, we have

$$(***) = E\left[|M^{\pi,\mathcal{A}}_{T} - M_T|^2\right] \le C\cdot\frac{1}{\mathcal{L}} + \hat{C}\cdot\frac{1}{K^2}.$$

As for term $(****)$, we have

$$\begin{aligned}
(****) &\overset{(i)}{=} E\left[\left|\sum_{j=1}^{\mathcal{J}}\left(v\left(T_j, X_{T_j}\right) - v\left(T_j, X^{\bar{\pi},\bar{\mathcal{A}}}_{T_j}\right) - E_{T_{j-1}}\left[v\left(T_j, X_{T_j}\right) - v\left(T_j, X^{\bar{\pi},\bar{\mathcal{A}}}_{T_j}\right)\right]\right)\right|^2\right]\\
&\overset{(ii)}{\le} L\sum_{j=1}^{\mathcal{J}} E\left[\left|X_{T_j} - X^{\bar{\pi},\bar{\mathcal{A}}}_{T_j}\right|^2\right] \overset{(iii)}{\le} \bar{L}|\bar{\pi}| \le \bar{L}|\pi| = \bar{L}\cdot\frac{1}{\mathcal{L}},
\end{aligned}$$

where L and $\bar{L}$ are some constants. Here equality (i) follows from (8), inequality (ii) follows from the Lipschitz continuity of $v(T_j, \cdot)$ and inequality (iii) follows from the mild continuity conditions that b, σ and J satisfy.

To this point the term left to estimate is $(**)$. Notice that, for $T_j \le t_l < T_{j+1}$ and $1 \le k \le \mathcal{K}$,

$$
\begin{aligned}
& E_{t_l}\left[(\Delta^\pi W_l)\left(v(T_{j+1}, X^{\bar{\pi},\bar{\mathcal{A}}}_{T_{j+1}}) - v\left(T_{j+1}, X_{T_{j+1}}\right)\right)\right]^2 \frac{1}{\Delta^\pi_l} \\
=& E_{t_l}\Bigg[(\Delta^\pi W_l)\Bigg(E_{t_{l+1}}\left[v(T_{j+1}, X^{\bar{\pi},\bar{\mathcal{A}}}_{T_{j+1}}) - v\left(T_{j+1}, X_{T_{j+1}}\right)\right] \\
& \quad - E_{t_l}\left[v(T_{j+1}, X^{\bar{\pi},\bar{\mathcal{A}}}_{T_{j+1}}) - v\left(T_{j+1}, X_{T_{j+1}}\right)\right]\Bigg)\Bigg]^2 \frac{1}{\Delta^\pi_l} \\
\le& E_{t_l}\Bigg[E_{t_{l+1}}\left[v(T_{j+1}, X^{\bar{\pi},\bar{\mathcal{A}}}_{T_{j+1}}) - v\left(T_{j+1}, X_{T_{j+1}}\right)\right]^2 - E_{t_l}\left[v(T_{j+1}, X^{\bar{\pi},\bar{\mathcal{A}}}_{T_{j+1}}) - v\left(T_{j+1}, X_{T_{j+1}}\right)\right]^2\Bigg],
\end{aligned}
$$

and similarly

$$
\begin{aligned}
& E_{t_l}\left[\tilde{P}\left([t_l, t_{l+1}] \times A_k\right)\left(v(T_{j+1}, X^{\bar{\pi},\bar{\mathcal{A}}}_{T_{j+1}}) - v\left(T_{j+1}, X_{T_{j+1}}\right)\right)\right]^2 \frac{1}{\mu\left([t_l, t_{l+1}] \times A_k\right)} \\
\le& E_{t_l}\Bigg[E_{t_{l+1}}\left[v(T_{j+1}, X^{\bar{\pi},\bar{\mathcal{A}}}_{T_{j+1}}) - v\left(T_{j+1}, X_{T_{j+1}}\right)\right]^2 - E_{t_l}\left[v(T_{j+1}, X^{\bar{\pi},\bar{\mathcal{A}}}_{T_{j+1}}) - v\left(T_{j+1}, X_{T_{j+1}}\right)\right]^2\Bigg].
\end{aligned}
$$

Hence,

$$
\begin{aligned}
(**) =& \sum_{j=0}^{\mathcal{J}-1} \sum_{T_j \le t_l < T_{j+1}} E\left[E_{t_l}\left[(\Delta^\pi W_l)\left(v(T_{j+1}, X^{\bar{\pi},\bar{\mathcal{A}}}_{T_{j+1}}) - v\left(T_{j+1}, X_{T_{j+1}}\right)\right)\right]^2 \frac{1}{\Delta^\pi_l}\right] \\
& + \sum_{j=0}^{\mathcal{J}-1} \sum_{T_j \le t_l < T_{j+1}} \sum_{k=1}^{\mathcal{K}} E\left[E_{t_l}\left[\tilde{P}\left([t_l, t_{l+1}] \times A_k\right)\left(v(T_{j+1}, X^{\bar{\pi},\bar{\mathcal{A}}}_{T_{j+1}}) - v\left(T_{j+1}, X_{T_{j+1}}\right)\right)\right]^2 \frac{1}{\mu\left([t_l, t_{l+1}] \times A_k\right)}\right] \\
\le& \, 2 \sum_{j=0}^{\mathcal{J}-1} E\left[\left|v(T_{j+1}, X^{\bar{\pi},\bar{\mathcal{A}}}_{T_{j+1}}) - v\left(T_{j+1}, X_{T_{j+1}}\right)\right|^2\right] \\
\le& \, 2L \sum_{j=1}^{\mathcal{J}} E\left[\left|X^{\bar{\pi},\bar{\mathcal{A}}}_{T_j} - X_{T_j}\right|^2\right] \le 2\bar{L}|\bar{\pi}| \le 2\bar{L}|\pi| = 2\bar{L} \cdot \frac{1}{\mathcal{L}}.
\end{aligned}
$$

Therefore,

$$
\begin{aligned}
& E\left[\max_{0 \le j \le \mathcal{J}} |\hat{M}^{\pi,\mathcal{A}}_{T_j} - \bar{M}_{T_j}|^2\right] \le 16[(*) + (**) + (***) + (****)] \\
\le& \, 16\left[\left(\mu\left([0,T] \times \mathrm{R}^d\right) + T\right)\epsilon + 2\bar{L} \cdot \frac{1}{\mathcal{L}} + C \cdot \frac{1}{\mathcal{L}} + \hat{C} \cdot \frac{1}{K^2} + \bar{L} \cdot \frac{1}{\mathcal{L}}\right] \\
\triangleq& \, \bar{C} \cdot \frac{1}{\mathcal{L}} + \bar{C}' \cdot \frac{1}{K^2} + \bar{C}'' \cdot \epsilon,
\end{aligned}
$$

where $\bar{C} = 16(C + 3\bar{L})$, $\bar{C}' = 16\hat{C}$, $\bar{C}'' = 16\left(\mu\left([0,T] \times \mathrm{R}^d\right) + T\right)$. $\square$